

PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION ONE WEEK
1 QUESTION 3 PAGES		TOTAL 100 POINTS

In this final take-home exam, we shall prove a well-known theorem of Scott that shows that Gödel's constructible universe cannot accommodate various types of large cardinals. Among other things, this is one of the reasons that the Axiom of Constructibility $V = L$ is not popular among set theorists.

The ultraproduct construction is a powerful tool that has only applications not only in model theory but also in algebra and analysis. In the case that the student is not familiar with this construction, we refer the student to any model theory book for a quick overview, e.g. *An Invitation to Mathematical Logic* by David Marker, which you can access through Springer Nature Link via the university's internet connection.

We begin by generalizing the ultrapower construction to take the ultrapower of the set-theoretic universe V . Let S be a non-empty set and $\mathcal{U} \subseteq \mathcal{P}(S)$ be an ultrafilter on S . Consider the proper class $\prod_{s \in S} V = \{f : f \text{ is a function with domain } S\}$ and the (class) relations $=^*$ and \in^* on $\prod_{s \in S} V$ given by

$$f =^* g \text{ if and only if } \{s \in S : f(s) = g(s)\} \in \mathcal{U}$$

$$f \in^* g \text{ if and only if } \{s \in S : f(s) \in g(s)\} \in \mathcal{U}$$

It is not difficult to check that $=^*$ is an equivalence relation on the class $\prod_{s \in S} V$. We wish to form the quotient set with respect to this equivalence relation. However, an equivalence class of $=^*$ would be a proper class and consequently, we cannot form the quotient class of equivalence classes in ZFC as it would be a class of proper classes. In order to get around this issue, we apply Scott's trick and define the equivalence class of an element $f \in \prod_{s \in S} V$ to be those functions of minimal \in -rank that are $=^*$ -equivalent to f , that is,

$$[f] = \left\{ g \in \prod_{s \in S} V : f =^* g \text{ and for all } h \in \prod_{s \in S} V \text{ with } f =^* h, \text{ we have } \text{rank}(g) \leq \text{rank}(h) \right\}$$

Under this definition, each equivalence class $[f]$ turns out to be a set. Therefore, we can form the definable class

$$\prod_{\mathcal{U}} V = \left\{ [f] : f \in \prod_{s \in S} V \right\}$$

The ultrapower of the set-theoretic universe V with respect to the ultrafilter \mathcal{U} is defined to be the structure $(\prod_{\mathcal{U}} V, \in^*)$. At this point, one should note that, perhaps surprisingly, unlike the ultrapower of a set structure, the ultrapower of the universe does not get "physically bigger" and we have $\prod_{\mathcal{U}} V \subseteq V$, that is, the ultrapower of the universe is indeed a subclass of the universe.

It turns out that Łoś's theorem holds for $(\prod_{\mathcal{U}} V, \in^*)$ as well, that is, for any formula $\varphi(x_1, \dots, x_n)$ in the language of set theory and any $[f_1], \dots, [f_n] \in \prod_{\mathcal{U}} V$, we have

$$\left(\prod_{\mathcal{U}} V, \in^* \right) \models \varphi([f_1], \dots, [f_n]) \text{ if and only if } \{s \in S : \varphi(f_1(s), \dots, f_n(s))\} \in \mathcal{U}$$

and that the map $\hat{j}_{\mathcal{U}} : V \rightarrow \prod_{\mathcal{U}} V$ sending each set x to the constant map $\hat{j}_{\mathcal{U}}(x) : S \rightarrow V$ with value x , that is, $\hat{j}_{\mathcal{U}}(x)(s) = x$ for every $s \in S$ is an elementary embedding. In other words, for any formula $\varphi(x_1, \dots, x_n)$ in the language of set theory and any sets a_1, \dots, a_n , we have

$$\varphi(a_1, \dots, a_n) \text{ if and only if } \left(\prod_{\mathcal{U}} V, \in^* \right) \models \varphi(\hat{j}_{\mathcal{U}}(a_1), \dots, \hat{j}_{\mathcal{U}}(a_n))$$

The ultrapower $(\prod_{\mathcal{U}} V, \in^*)$ is called *well-founded* if and only if

- every non-empty set $X \subseteq \prod_{\mathcal{U}} V$ has an \in^* -minimal element, and
- $\{[g] \in \prod_{\mathcal{U}} V : [g] \in^* [f]\}$ is a set for every $[f] \in \prod_{\mathcal{U}} V$.

Not every ultrapower is necessarily well-founded. Indeed, we have an exact characterization of well-founded ultrapowers in terms of the properties of the ultrafilter used. Recall that an ultrafilter is called κ -complete if it is closed under intersection of less than κ many sets.

- a) Show that $(\prod_{\mathcal{U}} V, \in^*)$ is well-founded if and only if \mathcal{U} is \aleph_1 -complete.
- b) Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} . Then, by Part (a), the ultrapower $(\prod_{\mathcal{U}} V, \in^*)$ is not well-founded. Explicitly find an infinite \in^* -decreasing sequence in $(\prod_{\mathcal{U}} V, \in^*)$ and explain why this does not contradict that $(\prod_{\mathcal{U}} V, \in^*) \models$ Axiom of Foundation.

Let κ be an uncountable regular cardinal. Suppose from now on that \mathcal{U} is a κ -complete ultrafilter. Then \mathcal{U} is \aleph_1 -complete trivially and hence, by Part (a), the ultrapower $(\prod_{\mathcal{U}} V, \in^*)$ is well-founded. In this case, we can apply Mostowski's collapsing theorem and consequently, the ultrapower $(\prod_{\mathcal{U}} V, \in^*)$ is isomorphic to a transitive proper class (M, \in) where the collapsing function $\pi : \prod_{\mathcal{U}} V \rightarrow M$ is such that $[f] \in^* [g]$ iff $\pi([f]) \in \pi([g])$ for all $[f], [g] \in \prod_{\mathcal{U}} V$. Now set $j_{\mathcal{U}} = \pi \circ \hat{j}_{\mathcal{U}}$. Then M is an inner model and $j_{\mathcal{U}} : V \rightarrow M$ is an elementary embedding. We shall now explore how the elementary embedding $j_{\mathcal{U}} : V \rightarrow M$ operates. For all we know, $j_{\mathcal{U}}$ could be the trivial elementary embedding.

- c) Prove that $j_{\mathcal{U}}(\alpha) = \alpha$ for all ordinals $\alpha < \kappa$.
- d) Using transfinite induction on \in -rank, prove that $j_{\mathcal{U}}(x) = x$ for every set $x \in V_{\kappa}$. Then figure out why you cannot simply prove that $j_{\mathcal{U}}(x) = x$ for every set x with the same method.

An uncountable cardinal κ is said to be *measurable* if there exists a κ -complete non-principal ultrafilter on κ . Measurable cardinals are a special type of large cardinals.

e) Prove that any measurable cardinal is inaccessible.

Consequently, the existence (or even, relatively consistency with ZFC) of a measurable cardinal cannot be proven in ZFC. On the other hand, there are various reasons to accept the existence of measurable cardinals that we shall not go into in this take-home final exam.

From now on, suppose that κ is a measurable cardinal and \mathcal{U} is a κ -complete non-principal ultrafilter on κ .

f) Prove that $j_{\mathcal{U}}(\kappa) > \kappa$.

In other words, $j_{\mathcal{U}} : V \rightarrow M$ is a *non-trivial* elementary embedding of the universe to an inner model in the case that \mathcal{U} is derived from a measurable cardinal. The fact that $j_{\mathcal{U}}(\kappa) > \kappa$ whenever κ is measurable, together with the previous observations, imply the following.

g) Prove that if there exists a measurable cardinal, then $V \neq L$.

It turns out that the existence of a non-trivial elementary embedding $j : V \rightarrow M$ implies the existence of a measurable cardinal as well, namely, the least ordinal α such that $j(\alpha) > \alpha$ is a measurable cardinal. However, we shall not cover this converse direction in this take-home final exam. Curious students are referred to Jech's classical book for a proof of this fact.

