PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 120 MINUTES
4 QUESTIONS ON 4 PAGES		TOTAL 50 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

***(12 pts)* 1.** Throughout this question, for any collection $\mathcal{E} \subseteq \mathcal{P}(\mathbb{R})$, let $\mathcal{M}(\mathcal{E})$ denote the σ-algebra **on** $\mathbb{R}$ generated by the collection $\mathcal{E}$. Fix a countable set $D \subseteq \mathbb{R}$. Prove that

$$\mathcal{M}(\{\{d\} : d \in D\}) = \mathcal{M}(\mathcal{P}(D)) \subsetneq \mathcal{B}(\mathbb{R})$$

(4+4+4 pts) **2.** Fix an enumeration $\mathbb{Q} = \{q_n\}_{n\in\mathbb{N}}$ of rational numbers without repetitions. Let $F : \mathbb{R} \to \mathbb{R}$ be given by

$$F(x) = \sum_{\substack{n\in\mathbb{N} \\ q_n \le x}} \frac{1}{2^{n+1}}$$

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$ where $\mu_F : \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ is the Lebesgue-Stieltjes measure associated to F.

a) Show that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$ is a probability space.

b) Show that $\int_{(a,b)} \chi_{\mathbb{Q}} \, d\mu_F > 0$ for every $a, b \in \mathbb{R}$ with $a < b$.

c) Describe a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ that is not integrable over $\mathbb{R}$ with respect to the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$. You only need to describe the function but need **not** prove your claim.

(13 pts) 3. Compute $\lim_{n\to\infty} \int_{[0,1]} n \sin\left(\frac{x}{n}\right) \frac{2}{1+x^2}\, d\mathbf{m}$. Explain each step in detail by referring to the relevant results.

***(13 pts)* 4.** Let $(X, \mathcal{M}, \mu)$ be a measure space and fix a map $f \in L^+(X, \mathcal{M}, \mu)$. Consider the map $\eta : \mathcal{M} \to [0, +\infty]$ given by

$$\eta(E) = \int_E f \, d\mu$$

You are **given** that η is a measure. Prove that, for every $g \in L^+(X, \mathcal{M}, \eta)$, we have

$$\int_X g \, d\eta = \int_X fg \, d\mu$$