PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 150 MINUTES
3 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

***(12+12+13+13 pts)* 1.** a) State Lebesgue's dominated convergence theorem, or, prove that $f_n \to f$ in L^1 implies $f_n \to f$ in measure.

In the remaining parts of this question, you will consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ with the measure ν given by

$$\nu(S) = \sum_{n \in S} \frac{1}{2^n}$$

b) Let $f : \mathbb{N} \to \mathbb{R}$ be a bounded function. Prove that f is integrable and

$$\int_{\mathbb{N}} f \, d\nu = \sum_{k=0}^{\infty} \frac{f(k)}{2^k}$$

Hint. Set $f_n(x) = f(x) \cdot \chi_{\{0,1,2,\ldots n\}}(x)$. Observe that $f_n \longrightarrow f$ pointwise. Then apply an appropriate theorem to get the result.

c) Let $f : \mathbb{N} \to \mathbb{R}$ be a function and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{R}$. Show that if $f_n \to f$ in measure, then $f_n \to f$ pointwise.

d) Let $g(x, k) = x^k$. Compute the integral

$$\int_{[1,3/2]\times\mathbb{N}} g(x, k)\ d(\mathbf{m} \times \nu)$$

in the product measure space $(\mathbb{R}\times\mathbb{N}, \mathcal{B}(\mathbb{R})\otimes\mathcal{P}(\mathbb{N}), \mathbf{m}\times\nu)$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$ and $\mathbf{m}$ is the Lebesgue measure. Explain each step of your computation by referring to the relevant theorems.

***(10+10 pts)* 2.** Consider the signed measure μ on the measurable space $([0, 2\pi], \mathcal{B}([0, 2\pi]))$ given by

$$\mu(A) = \int_A \sin(x)\ d\mathbf{m}$$

a) Find the Hahn decomposition (P, N) of the measure space $([0, 2\pi], \mathcal{B}([0, 2\pi]), \mu)$.

b) Let (μ^+, μ^-) be the Jordan decomposition of the measure μ. Show that $\mu^+ \ll \mathbf{m}$ and find the Radon-Nikodym derivative $\dfrac{d\mu^+}{d\mathbf{m}}$.

(10+10+10 pts) **3.** Consider the set

$$\begin{aligned} A &= \left\{x \in [0,1] : \ x = (0.a_1a_2a_3\dots)_2 \text{ where } a_n \in \{0,1\} \text{ and } \ a_{2n} = 1 \text{ for every } n \in \mathbb{N}^+\right\} \\ &= \{x \in [0,1] : \ \text{the binary expansion of } x \text{ contains 1 at its even numbered digits}\} \\ &= \bigcap_{n \in \mathbb{N}^+} \{x \in [0,1] : \ \text{the binary expansion of } x \text{ contains 1 at its digits } a_2, a_4, \dots, a_{2n}\} \end{aligned}$$

a) Show that $\mathbf{m}(A) = 0$.

b) Show that $[0,1] \setminus A$ is open. Using this, conclude that $\chi_A(x)$ is continuous at every point in $[0,1] \setminus A$.

c) Show that $\chi_A(x)$ is Riemann integrable over $[0,1]$ and find $\displaystyle\int_0^1 \chi_A(x)dx$ by justifying each step of your computation.