M E T U Department of Mathematics

Math 497 Hilbert Space Techniques Fall 2025 Midterm II 20 December 2025 13:40		
F U L L N A M E	S T U D E N T I D	DURATION 140 MINUTES
4 QUESTIONS ON 4 PAGES		TOTAL 100(+20) POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

In all questions, the underlying field is $\mathbb{K} = \mathbb{C}$.

***(12+12+12 pts)* 1.** Consider the Volterra operator $V : L^2(0,1) \to L^2(0,1)$ given by

$$(Vf)(t) = \int_0^t f(x)dx$$

for all $f \in L^2(0,1)$. You are given that V is a linear operator.

a) Show that V is a bounded operator and that $\|V\| \leq 1/\sqrt{2}$.

b) Show that V is a Hilbert-Schmidt operator.

c) Find the adjoint operator V^*.

(12+12+12 pts) **2.** Let K be a compact Hermitian operator on a Hilbert space H.
a) Show that K is the limit of finite rank operators in $L(H)$.

b) Show that the closure of the range of K is a separable Hilbert space.

c) Show that if $K^n = \mathbf{0}$ for some $n \in \mathbb{Z}^+$, then $K = \mathbf{0}$.

(12+12 pts) **3.** a) Show that any infinite orthonormal sequence in a Hilbert space weakly converges to the zero vector.

b) Show that, in a finite-dimensional Hilbert space, any weakly convergent sequence is strongly convergent.

(12+12 pts) **4.** Let H be a Hilbert space. You are given that the set $L(H)$ of bounded linear operators on H forms a ring, together with pointwise addition of operators as its addition and the composition of operators as its multiplication.

a) Show that the set $K(H)$ of compact operators on H is a two-sided ideal of $L(H)$.

b) Consider the quotient ring $\mathcal{Q}(H) = L(H)/K(H)$. Show that the map $\iota_U : \mathcal{Q}(H) \to \mathcal{Q}(H)$ given by $\iota_U(X + K(H)) = UXU^* + K(H)$ is a ring automorphism of $\mathcal{Q}(H)$ for any unitary operator $U \in L(H)$.