M E T U Department of Mathematics

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<tr><td>F U L L N A M E</td><td>S T U D E N T I D</td><td>DURATION
140 MINUTES</td></tr>
<tr><td colspan="2">5 QUESTIONS ON 4 PAGES</td><td>TOTAL 100 POINTS</td></tr>
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By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

In Question 1 and Question 2, the underlying field is $\mathbb{K} = \mathbb{R}$, and in Question 3, the underlying field is $\mathbb{K} = \mathbb{C}$.

(4+8+4+8 pts) **1.** a) State the parallelogram law: Let V be an inner product space over $\mathbb{K}$. Then, for any vectors $x, y \in V$, we have ...

Let $C[0,1] = \{f : [0,1] \to \mathbb{K} \ \mid \ f \text{ is continuous}\}$.

b) Show that the supremum norm $\|\cdot\|_\infty : C[0,1] \to \mathbb{R}$ given by $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$ is not induced by any inner product on $C[0,1]$.

c) State the Jordan-von Neumann theorem: Let V be a normed space over $\mathbb{K}$. Then the norm $\|\cdot\|_V$ is induced by an inner product if and only if ...

d) Consider the norm $\|\cdot\| : C[0,1] \to \mathbb{R}$ given by

$$\|f\| = \sqrt{\int_0^1 (1 + x^2)(f(x))^2 dx}$$

Show that $\|\cdot\|$ is induced by an inner product on $C[0,1]$ and find this inner product explicitly.

(8×8 pts) **2.** Consider the inner product space $\ell_2 = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$ together with its standard inner product

$$\langle (a_n)_{n\in\mathbb{N}} \mid (b_n)_{n\in\mathbb{N}} \rangle = \sum_{n=0}^{\infty} a_n b_n$$

Set $S = \{(a_n)_{n\in\mathbb{N}} \in \ell_2 : a_n \geq 0 \text{ for all } n \in \mathbb{N}\}$ and $\mathbf{x} = \left(\frac{(-1)^n}{n+1}\right)_{n\in\mathbb{N}} = \left(1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots\right)$.

a) Show that S is a closed convex subset of ℓ_2.

b) Determine whether or not there exists a unique vector in S that is closest to $\mathbf{x}$.

For Part (c) **only**, we endow ℓ_2 with the supremum norm given by $\|(a_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$. You are given that, with respect to this norm, S is still a closed convex subset of ℓ_2.

c) Determine whether or not there exists a unique vector in S that is closest to $\mathbf{x}$ with respect to the norm $\|\cdot\|_\infty$.

In the rest of this question, we continue endowing ℓ_2 and its subsets with its standard inner product and the induced norm $\|\cdot\|_2$. Recall that, for each $n \in \mathbb{N}$, $\mathbf{e}_n = (\delta_{kn})_{k\in\mathbb{N}}$ denotes the standard basis vector whose n-th entry 1 and whose other entries are 0. Consider the vector subspace

$$\mathcal{H} = \{(a_n)_{n\in\mathbb{N}} \in \ell_2 : \ a_{2n} = a_{2n+1} \text{ for each } n \in \mathbb{N}\}$$

d) Show that $\mathcal{H}$ is a Hilbert space.

e) Find a complete orthonormal sequence for the Hilbert space $\mathcal{H}$.

f) Find an inner product space isomorphism, that is, a unitary map, $\varphi : \ell_2 \to \mathcal{H}$. For this part of the question, it suffices to only define the map φ without actually proving that it is an isomorphism.

In Part (g) and (h), you shall consider $c_{00} = \{(a_n)_{n\in\mathbb{N}} \in \ell_2 : \exists m \in \mathbb{N}\ \forall n \geq m\ a_n = 0\}$ as a vector subspace of ℓ_2 with the restriction of the standard inner product and norm of ℓ_2 to this subspace.

A corollary of the Hahn-Banach theorem states the following: Let $(E, \|\cdot\|_E)$ be a normed space, $W \subseteq E$ be a subspace and $f : W \to \mathbb{K}$ be a continuous linear functional. Then there exists a continuous linear extension $F : E \to \mathbb{K}$ of f.

g) Let $f : c_{00} \to \mathbb{K}$ be a bounded linear functional. Show that there exists $(b_n)_{n\in\mathbb{N}} \in \ell_2$ such that $f\left((a_n)_{n\in\mathbb{N}}\right) = \langle (a_n)_{n\in\mathbb{N}} \mid (b_n)_{n\in\mathbb{N}} \rangle$ for all $(a_n)_{n\in\mathbb{N}} \in c_{00}$.

h) Construct a bounded linear functional $f : c_{00} \to \mathbb{K}$ such that there exists no $(b_n)_{n\in\mathbb{N}} \in c_{00}$ such that $f\left((a_n)_{n\in\mathbb{N}}\right) = \langle (a_n)_{n\in\mathbb{N}} \mid (b_n)_{n\in\mathbb{N}} \rangle$ for all $(a_n)_{n\in\mathbb{N}} \in c_{00}$. Explain why this does not contradict the Riesz-Fréchet representation theorem.

***(5+2+5 pts)* 3.** Recall the sequence $\left(\frac{e^{inx}}{\sqrt{2\pi}}\right)_{n=-\infty}^{\infty}$ is a complete orthonormal sequence for the Hilbert space $L^2(-\pi, \pi)$.

a) Compute the n-th Fourier coefficient of $f(x) = x^2$ with respect to $\left(\frac{e^{inx}}{\sqrt{2\pi}}\right)_{n=-\infty}^{\infty}$.

b) Complete the following statement of Parseval's formula appropriately: Let $f, g \in L^2(-\pi, \pi)$ have the Fourier series $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and $g(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx = \ldots\ldots\ldots$$

c) Show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.