METU Department of Mathematics

Math 497 Hilbert Space Techniques Fall 2025 Final Exam 14 January 2026 09:30		
FULL NAME	STUDENT ID	DURATION 140 MINUTES
3 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

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In all questions, the underlying field is $\mathbb{K} = \mathbb{C}$.

***(5+15+5+10 pts)* 1.** In this question, you shall be guided through a proof of that boundedness implies weak sequential compactness in a separable Hilbert space and you shall fill in the gaps.

Theorem. Let $\mathcal{H}$ be a separable Hilbert space and $(x_n)_{n\in\mathbb{N}}$ be bounded sequence. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}^+}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ is weakly convergent.

Proof. Since $\mathcal{H}$ is separable, we can fix a countable dense subset $(a_n)_{n\in\mathbb{N}}$.

a) Prove that the sequence $\big(\langle x_n | a_0\rangle\big)_{n\in\mathbb{N}}$ is bounded in $\mathbb{K}$.

Since the sequence $\big(\langle x_n | a_0\rangle\big)_{n\in\mathbb{N}}$ is bounded in $\mathbb{K}$, by the Bolzano-Weierstrass theorem,

- there exists a subsequence $(x_n^0)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\big(\langle x_n^0 | a_0\rangle\big)_{n\in\mathbb{N}}$ converges and hence, is bounded.

Now, as in Part (a), the sequence $\big(\langle x_n^0 | a_1\rangle\big)_{n\in\mathbb{N}}$ is bounded in $\mathbb{K}$ and hence, as before,

- there exists a subsequence $(x_n^1)_{n\in\mathbb{N}}$ of $(x_n^0)_{n\in\mathbb{N}}$ such that $\big(\langle x_n^1 | a_1\rangle\big)_{n\in\mathbb{N}}$ converges and hence, is bounded.

Continuing in this manner, for each $k \in \mathbb{N}$, we can construct a subsequence $(x_n^{k+1})_{n\in\mathbb{N}}$ of $(x_n^k)_{n\in\mathbb{N}}$ such that $\big(\langle x_n^k | a_k\rangle\big)_{n\in\mathbb{N}}$ converges and hence, is bounded. Let $(z_n)_{n\in\mathbb{N}}$ be the "diagonal" sequence given by

$$z_n = x_n^n$$

for all $n \in \mathbb{N}$. We claim that the subsequence $(z_n)_{n\in\mathbb{N}}$ is weakly convergent.

b) Using an ϵ-argument together with the density of $(a_n)_{n\in\mathbb{N}}$, show that the sequence $\big(\langle z_n | w\rangle\big)_{n\in\mathbb{N}}$ is Cauchy and hence, converges for every $w \in \mathcal{H}$.

Consider the map $L : \mathcal{H} \to \mathbb{K}$ given by

$$L(w) = \lim_{n\to\infty} \langle w \mid z_n \rangle$$

for all $w \in \mathcal{H}$. Clearly, inner product being linear in the first component, L is a linear functional.
c) Show that L is bounded.

d) Show that there exists $x \in \mathcal{H}$ such that $z_n \rightharpoonup x$.

This completes the proof that $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence. ■
(10+10+15 pts) **2.** In this question, you shall be guided through a proof of the Banach-Saks theorem for separable Hilbert spaces and you shall fill in the gaps.

Theorem. (Banach-Saks) Let $\mathcal{H}$ be a separable Hilbert space. Let $(x_n)_{n\in\mathbb{N}^+}$ be a bounded sequence in $\mathcal{H}$. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}^+}$ such that the Césaro limit of $(x_{n_k})_{k\in\mathbb{N}^+}$ exists, that is, the sequence

$$\left(\frac{x_{n_1} + \cdots + x_{n_k}}{k}\right)_{k\in\mathbb{N}^+}$$

converges to some $x \in \mathcal{H}$.

Proof. Using Question 1, by replacing it with an appropriate subsequence if necessary, we may assume without loss of generality that the sequence $(x_n)_{n\in\mathbb{N}^+}$ is weakly convergent, say, $x_n \rightharpoonup x$ for some $x \in \mathcal{H}$. We claim that x is the Césaro limit of a subsequence. Our idea is to recursive choose a sequence $(n_k)_{k\in\mathbb{N}^+}$ such that the statement

$$*_k : \quad |\langle x_{n_i} - x | x_{n_k} - x\rangle| \leq \frac{1}{k} \quad \text{for each } i = 1, \ldots, k-1$$

holds for all $k \geq 2$. Set $n_1 = 1$. Let $k \geq 1$ and assume that $n_1, \ldots, n_k \in \mathbb{N}^+$ have been chosen so that the statements $*_2, \ldots, *_k$ hold. We shall choose $n_{k+1} \in \mathbb{N}^+$ so that the statement $*_{k+1}$ holds as well.
a) Show that, for each $i = 1, \ldots, k$, we have $\lim\limits_{m\to\infty} \langle x_{n_i} - x | x_m - x\rangle = 0$.

b) Choose $n_{k+1} \in \mathbb{N}^+$ so that $*_{k+1}$ holds, that is

$$|\langle x_{n_i} - x | x_{n_{k+1}} - x \rangle| \leq \frac{1}{k+1} \quad \text{for each } i = 1, \dots, k$$

It now follows from Part (b) that such a recursive construction of $(n_k)_{k\in\mathbb{N}^+}$ is possible. Since $(x_n)_{n\in\mathbb{N}^+}$ is bounded, so is $(x_n - x)_{n\in\mathbb{N}^+}$, say, $\|x_n - x\| \leq M$ for every $n \in \mathbb{N}$.

c) Let $k \in \mathbb{N}^+$. Show that

$$\left\| \sum_{i=1}^{k} (x_{n_i} - x) \right\|^2 \leq 2k + kM^2$$

and deduce that $\lim\limits_{k\to\infty} \dfrac{x_{n_1} + \cdots + x_{n_k}}{k} = x$.

This completes the proof of the theorem. ■

(10+10+10 pts) **3.** Determine whether the following statements are true or false. If the statement is true, prove the statement. If the statement is false, disprove the statement by providing a counterexample.

a) **TRUE OR FALSE:** For every Hilbert space H and every subspace $M \subseteq H$, we have $\left(M^{\perp}\right)^{\perp} = M$.

b) **TRUE OR FALSE:** For every Hilbert space H and every bounded linear operator $T \in L(H)$, if $T^2 = \mathbf{0}$, then T is compact.

c) **TRUE OR FALSE:** For every Hilbert space H and every bounded linear operator $T \in L(H)$, we have $\sigma(T^*) = \{\overline{z} \in \mathbb{K} : z \in \sigma(T)\}$.