PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 120+ϵ MINUTES
3 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

Throughout this exam, we shall work with normed spaces over the field $\mathbb{K} = \mathbb{R}$. In your solutions, you can quote all facts that we have learned in class (or other classes such as MATH349) and in lecture notes without proof. However, if you wish to use a (non-trivial) fact that is not covered in class, then you are supposed to prove it.

***(6×12 pts)* 1.** For each $k \in \mathbb{N}$, consider the subspace

$$\mathcal{X}_k = \left\{ (a_n)_{n\in\mathbb{N}} \in \ell_1 : \sum_{n=0}^{\infty} a_n x^n \text{ is a polynomial of degree at most } k \right\}$$

of the Banach space $(\ell_1, \|\cdot\|_1)$.

a) Show that $\mathcal{X}_k$ is nowhere dense for every $k \in \mathbb{N}$.

b) Set $\mathcal{X} = \bigcup_{k\in\mathbb{N}} \mathcal{X}_k$. Using the Baire category theorem, show that $(\mathcal{X}, \|\cdot\|_1)$ is not a Banach space.

c) For each $m \in \mathbb{N}$, consider the shift map $\sigma_m : \mathcal{X} \to \mathcal{X}$ given by

$$\sigma_m\left((a_n)_{n\in\mathbb{N}}\right) = \left(m \cdot a_{n+1}\right)_{n\in\mathbb{N}}$$

Show that σ_m is bounded.

d) For each $m \in \mathbb{N}^+$, set $\sigma_m^{(m)} = \underbrace{\sigma_m \circ \cdots \circ \sigma_m}_{m \text{ times}}$. Show that, for each $\mathbf{a} \in \mathcal{X}$, we have that $\sup_{m\in\mathbb{N}^+} \left\|\sigma_m^{(m)}(\mathbf{a})\right\|_1 < \infty$.

e) Show that $\sup_{m\in\mathbb{N}^+} \left\|\sigma_m^{(m)}\right\| = \infty$.

f) Explain why Part (d) and Part (e) together do not contradict the uniform boundedness principle.

(10+10 pts) **2.** In this question, consider the Banach space $(\ell_\infty, \|\cdot\|_\infty)$ and its closed subspace

$$S = \{(a_n)_{n\in\mathbb{N}} \in \ell_\infty : \ a_{2n} = 0 \text{ for all } n \in \mathbb{N}\}$$

a) Show that the Banach spaces $(\ell_\infty/S, \|\cdot\|_q)$ and $(\ell_\infty, \|\cdot\|_\infty)$ are isomorphic, where $\|\cdot\|_q$ is the quotient norm.

b) Let $Q \subseteq \ell_\infty$ be an algebraic complement of S in ℓ_∞, that is, Q is a subspace such that $\ell_\infty = S \bigoplus Q$. Prove that $\ell_\infty = S \bigoplus_{top} Q$, that is, S and Q are topologically complementary subspaces.

(8 pts) **3.** Consider the quotient vector space ℓ_∞ / c_{00} and the map $\|\cdot\|_q : \ell_\infty / c_{00} \to \mathbb{R}$ given by

$$\|\mathbf{x} + c_{00}\|_q = \inf_{\mathbf{y} \in c_{00}} \|\mathbf{x} - \mathbf{y}\|_\infty$$

for all $\mathbf{x} \in \ell_\infty$. Show that $\|\cdot\|_q$ is **not** a norm by explicitly finding a non-zero element $\mathbf{a} + c_{00} \in \ell_\infty / c_{00}$ such that $\|\mathbf{a} + c_{00}\|_q = 0$.