METU Department of Mathematics, Math 452, Midterm I, May 2, 2024

<table>
<tr><td colspan="3">PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS</td></tr>
<tr><td>F U L L N A M E</td><td>S T U D E N T I D</td><td>DURATION
120+ϵ MINUTES</td></tr>
<tr><td colspan="2">2 QUESTIONS ON 4 PAGES</td><td>TOTAL 100 POINTS</td></tr>
</table>

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

Throughout this exam, we shall work with normed spaces over the field $\mathbb{K} = \mathbb{R}$. In your solutions, you can quote all facts that we have learned in class (or other classes such as MATH349) and in lecture notes without proof. However, if you wish to use a (non-trivial) fact that is not covered in class, then you are supposed to prove it.

(7×10 pts) **1.** Consider the subspace

$$\mathcal{X} = \left\{ (a_n)_{n\in\mathbb{N}} \in \ell_1 : \sum_{n=0}^{\infty} a_n x^n \text{ converges absolutely for all } x \in \mathbb{R} \right\}$$

of the vector space ℓ_1. In this question, we shall endow $\mathcal{X}$ first with the norm $\|\cdot\|_1$ and then with the norm $\|\cdot\|_\infty$.

a) Show that $\mathcal{X}$ is dense in the normed space $(\ell_1, \|\cdot\|_1)$.

We shall show that every open ball in ℓ_1 contains an element of $\mathcal{X}$. Let $\mathbf{a} \in \ell_1$ and $\epsilon > 0$. Since $\sum_{n=0}^{\infty} |a_n| < \infty$, there exists some $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |a_n| < \epsilon$. Consider the sequence $\mathbf{x} = (a_0, \dots, a_n, 0, 0, \dots)$. Then the power series corresponding to $\mathbf{x}$ is a polynomial and hence converges everywhere absolutely. Consequently, $\mathbf{x} \in \mathcal{X}$. Moreover

$$\|\mathbf{a} - \mathbf{x}\|_1 = \sum_{n=N+1}^{\infty} |a_n| < \epsilon$$

and so $\mathbf{x} \in B_{\|\cdot\|}(\mathbf{a}, \epsilon)$. Hence $\mathcal{X}$ is dense in ℓ_1.

b) Show that the normed space $(\mathcal{X}, \|\cdot\|_1)$ is not a Banach space.

By Part (a), we know that $\overline{\mathcal{X}} = \ell_1$. Observe that the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ has radius of convergence 1 and so $\left(1, \frac{1}{4}, \frac{1}{9}, \dots\right) \notin \mathcal{X}$. However, $\left(1, \frac{1}{4}, \frac{1}{9}, \dots\right) \in \ell_1$. Therefore, $\mathcal{X} \neq \ell_1$ and so $\mathcal{X} \neq \overline{\mathcal{X}}$. Since $\mathcal{X}$ is not closed in ℓ_1, $(\mathcal{X}, \|\cdot\|_1)$ is not a Banach space.

c) Fix $r \in \mathbb{R}$. Consider the linear functional $\varphi_r : \mathcal{X} \to \mathbb{R}$ given by

$$\varphi_r\left((a_n)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty} a_n r^n = a_0 + a_1 r + a_2 r^2 + \dots$$

on the normed space $(\mathcal{X}, \|\cdot\|_1)$. Show that φ_r is bounded if and only if $|r| \leq 1$.

To prove the left-to-right direction, assume that φ_r is bounded. Then there exists $M > 0$ such that $|\varphi_r(\mathbf{x})| \leq M \|\mathbf{x}\|_1$ for all $\mathbf{x} \in \mathcal{X}$. If $|r| > 1$, then $\lim_{n\to\infty} |r|^n = +\infty$ and hence the inequality

$$|\varphi_r(\mathbf{e}_n)| = |r^n| = |r|^n \leq M \|\mathbf{e}_n\|_1 = M$$

would be violated for a sufficiently large n, where $\mathbf{e}_n = (\delta_{in})_{i\in\mathbb{N}}$. Thus $|r| \leq 1$.

To prove the right-to-left direction, assume that $|r| \leq 1$. Then, for any $\mathbf{x} \in \mathcal{X}$, we have

$$|\varphi_r(\mathbf{x})| = \left|\sum_{n=0}^{\infty} x_n r^n\right| \leq \sum_{n=0}^{\infty} |x_n r^n| = \sum_{n=0}^{\infty} |x_n|\,|r|^n \leq \sum_{n=0}^{\infty} |x_n| = \|\mathbf{x}\|_1$$

and hence φ_r is bounded.

d) Find $\|\varphi_{0.452}\|$.

By the argument in Part (a), we know that $|\varphi_r(\mathbf{x})| \leq \|\mathbf{x}\|_1$ for all $\mathbf{x} \in \mathcal{X}$ and so

$$\|\varphi_{0.452}\| = \sup_{\|\mathbf{x}\|_1 \leq 1} |\varphi_r(\mathbf{x})| \leq \sup_{\|\mathbf{x}\|_1 \leq 1} \|\mathbf{x}\|_1 = 1$$

Moreover, since $\|(1,0,0,\dots)\|_1 = 1$, we have

$$\varphi_{0.452}(1,0,0,\dots) = 1 \leq \sup_{\|\mathbf{x}\|_1 \leq 1} |\varphi_r(\mathbf{x})| = \|\varphi_{0.452}\|$$

Hence $\|\varphi_{0.452}\| = 1$.

e) Fix $r \in \mathbb{R}$. Consider the linear functional $\psi_r : \mathcal{X} \to \mathbb{R}$ given by

$$\psi_r\left((a_n)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty} a_n r^n = a_0 + a_1 r + a_2 r^2 + \dots$$

on the normed space $(\mathcal{X}, \|\cdot\|_\infty)$. Show that ψ_r is bounded if and only if $|r| < 1$.

To prove the left-to-right direction, assume that ψ_r is bounded. Then there exists $M > 0$ such that $|\psi_r(\mathbf{x})| \leq M \|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in \mathcal{X}$. If $|r| > 1$, then the inequality

$$\left|\psi_r\left(\sum_{i=0}^{n} \mathbf{e}_i\right)\right| = |1 + r + r^2 + \dots + r^n + 0 + 0 + \dots| = \left|\frac{r^{n+1} - 1}{r - 1}\right| \leq M \left\|\sum_{i=0}^{n} \mathbf{e}_i\right\|_\infty = M$$

would be violated for a sufficiently large n, where $\mathbf{e}_n = (\delta_{in})_{i\in\mathbb{N}}$. Thus $|r| \leq 1$.

To prove the right-to-left direction, assume that $|r| < 1$. Then, for any $\mathbf{x} \in \mathcal{X}$, we have

$$|\psi_r(\mathbf{x})| = \left|\sum_{n=0}^{\infty} x_n r^n\right| \leq \sum_{n=0}^{\infty} |x_n r^n| = \sum_{n=0}^{\infty} |x_n|\,|r|^n \leq \sum_{n=0}^{\infty} \|\mathbf{x}\|_\infty |r|^n \leq \frac{\|\mathbf{x}\|_\infty}{1 - |r|}$$

and hence φ_r is bounded.

f) Find $\|\psi_{0.452}\|$.

By the argument in Part (a), we know that $|\psi_r(\mathbf{x})| \leq \frac{\|\mathbf{x}\|_\infty}{1-0.452}$ for all $\mathbf{x} \in \mathcal{X}$ and so

$$\|\psi_{0.452}\| = \sup_{\|\mathbf{x}\|_\infty \leq 1} |\psi_{0.452}(\mathbf{x})| \leq \sup_{\|\mathbf{x}\|_\infty \leq 1} \frac{\|\mathbf{x}\|_\infty}{1 - 0.452} = \frac{1}{0.548}$$

Moreover, since $\|\sum_{i=0}^{n} \mathbf{e}_i\|_\infty = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left|\psi_{0.452}\left(\sum_{i=0}^{n} \mathbf{e}_i\right)\right| = \lim_{n\to\infty} \frac{1 - 0.452^{n+1}}{1 - 0.452} = \frac{1}{0.548}$$

we must have $\frac{1}{0.548} \leq \sup_{\|\mathbf{x}\|_\infty \leq 1} |\psi_r(\mathbf{x})| = \|\psi_{0.452}\|$. Hence $\|\psi_{0.452}\| = \frac{1}{0.548}$.

g) If exists, provide a Schauder basis for the normed space $(\mathcal{X}, \|\cdot\|_\infty)$. If not, state that such a Schauder basis does not exist. For this part of the question only, you need **not** justify your answer.

The sequence $(\mathbf{e}_n)_{n\in\mathbb{N}}$, where $\mathbf{e}_n = (\delta_{in})_{i\in\mathbb{N}}$, is a Schauder basis for $(\mathcal{X}, \|\cdot\|_\infty)$.

***(3×10 pts)* 2.** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces and $T \in L(E, F)$. We define the *adjoint* of T to be the bounded linear map $T^* : F' \to E'$ given by $T^*(\varphi) = \varphi \circ T$. In other words, we have

$$T^*(\varphi)(e) = (\varphi \circ T)(e) = \varphi(T(e))$$

for all $\varphi \in F'$ and $e \in E$.

a) Let $\varphi \in F'$ and consider the bounded linear functional $T^*(\varphi) : E \to \mathbb{R}$. Write down the definition of $\|T^*(\varphi)\|$.

$\|T^*(\varphi)\| = \sup_{\|e\|_E \le 1} |T^*(\varphi)(e)| = \sup_{\|e\|_E \le 1} |\varphi(T(e))|$

b) Show that $\|T^*\| \le \|T\|$.

$$\|T^*\| = \sup_{\substack{\varphi\in F' \\ \|\varphi\|\le 1}} \|T^*(\varphi)\| = \sup_{\substack{\varphi\in F' \\ \|\varphi\|\le 1}} \sup_{\|e\|_E\le 1} |\varphi(T(e))| \le \sup_{\substack{\varphi\in F' \\ \|\varphi\|\le 1}} \sup_{\|e\|_E\le 1} \|\varphi\| \, \|T(e)\|_F \le \sup_{\substack{\varphi\in F' \\ \|\varphi\|\le 1}} \sup_{\|e\|_E\le 1} \|\varphi\| \, \|T\| \, \|e\|_E \le \|T\|$$

c) Using (one of the corollaries of) the Hahn-Banach theorem, show that $\|T^*\| \ge \|T\|$.

Observe that the inequality $\|T^*\| \ge \|T\|$ trivially holds when T is the zero functional since we have $\|T^*\| = \|T\| = 0$ in this case.

Suppose that T is not the zero functional. Then there exists $e \in E$ such that $T(e) \ne \mathbf{0}_F$. Set $\hat{e} = \frac{e}{\|e\|_E}$. Applying Corollary 20 in the lecture notes with the non-zero vector $T(\hat{e})$ in the space $(F, \|\cdot\|_F)$, we know that there exists a bounded linear functional $\psi \in F'$ such that $\|\psi\| = 1$ and $\psi(T(\hat{e})) = \|T(\hat{e})\|_F$. But then, we have

$$T^*(\psi)(\hat{e}) = \psi(T(\hat{e})) = \|T(\hat{e})\|_F = \frac{\|T(e)\|_F}{\|e\|_E}$$

Observe that $\|\hat{e}\|_E = 1$. It follows that

$$\|T^*\| = \sup_{\substack{\varphi\in F' \\ \|\varphi\|\le 1}} \sup_{\|x\|_E\le 1} |\varphi(T(x))| \ge |\psi(T(\hat{e}))| = \frac{\|T(e)\|_F}{\|e\|_E}$$

which means $\|T(e)\|_F \le \|T^*\| \, \|e\|_E$. By Proposition 8 in the lecture notes, we now have $\|T^*\| \ge \|T\|$.