PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION $120+\epsilon$ MINUTES
2 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

In your solutions, you can quote all facts that we have learned in class (or other classes such as MATH349) and in lecture notes without proof. However, if you wish to use a (non-trivial) fact that is not covered in class, then you are supposed to prove it.

***(7×10 pts)* 1.** The following is a classical theorem of Banach.

Theorem. Every Banach space $(E, \|\cdot\|)$ is isometrically isomorphic to a closed subspace of the Banach space $(C(X), \|\cdot\|_\infty)$ for some compact Hausdorff space X.

In this question, our aim is to prove this theorem with guidance through the parts of the question.

Proof. Let $X = B_{E'} = \{f \in E' : \ \|f\| \leq 1\}$ endowed with the (subspace topology induced from the) weak-$*$ topology.

a) Explain why X is a compact Hausdorff space by referring to the relevant theorems.

Consider the map $\varphi : E \to C(X)$ given by $e \mapsto \varphi_e$, where $\varphi_e : X \to \mathbb{K}$ is defined by $\varphi_e(f) = f(e)$ for all $f \in X$. Our candidate linear isometry is φ. We first need to check that each φ_e is continuous so that $\varphi_e \in C(X)$ and consequently, the map φ is well-defined.

b) Let $e \in E$. Show that $\varphi_e : X \to \mathbb{K}$ is a continuous map.

Warning. Keep in mind that the topology of X is not the norm topology but rather the weak-$*$ topology. Indeed X is not even a normed space.

Now that we know that φ is well-defined, let us check that φ is indeed a linear map.

c) Show that $\varphi : E \to C(X)$ is a linear map.

We shall next check that φ is an isometry. Let $e \in E$.

d) Show that $\|\varphi_e\|_\infty \le \|e\|$.

e) Show that there exists $f \in X$ such that $f(e) = \|e\|$ by referring to the relevant theorems.

f) Deduce that $\|\varphi_e\|_\infty = \|e\|$ and hence φ is an isometry.

Now that we know that φ is an isometry, it suffices to...

g) Show that $\varphi(E)$ is closed in $C(X)$.

Consequently, $(E, \|\cdot\|)$ is isometrically isomorphic to the closed subspace $(\varphi(E), \|\cdot\|_\infty)$ of the Banach space $(C(X), \|\cdot\|_\infty)$ via the map φ. $\blacksquare$

***(10+10+10 pts)* 2.** Let $(E, \|\cdot\|)$ be an infinite-dimensional normed space.

a) Let $U \subseteq E$ be a weakly-open set such that $\mathbf{0} \in U$. Show that there exists an infinite-dimensional norm-closed subspace C with $C \subseteq U$. (**Hint.** Pick an appropriate neighborhood basis element of $\mathbf{0}$ and try using the data encoded in that.)

b) Show that the closed unit ball $B_E = \{e \in E : \ \|e\| \leq 1\}$ has empty interior in the weak topology of E.

c) Consider the topological space (E', τ_{w^*}) where τ_{w^*} denotes the weak-$*$ topology. Show that E' is meager in itself with respect to the weak-$*$ topology. (**Hint.** Consider the scaled closed balls $nB_{E'}$ where $B_{E'}$ is the unit closed ball in the norm topology and $n \in \mathbb{N}^+$.)