PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 120+ϵ MINUTES
4 QUESTIONS ON 4 PAGES		TOTAL 120 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

Throughout this exam, we shall work with normed spaces over the field $\mathbb{K} = \mathbb{R}$. In your solutions, you can quote all facts that we have learned in class (or other classes such as MATH349) without proof. However, if you wish to use a (non-trivial) fact that is not covered in class, then you are supposed to prove it.

***(12+12 pts)* 1.** Consider the Banach space $(\ell_\infty, \|\cdot\|_\infty)$. For each $k \in \mathbb{N}$, define

$$W_k = \{\mathbf{x} \in \ell_\infty : \ \forall n \geq k \ \ x_n = 0\}$$

a) Prove that the proper subspace W_k is nowhere dense for all $k \in \mathbb{N}$.

b) Prove that $c_{00} = \{\mathbf{x} \in \ell_\infty : \ \exists k \in \mathbb{N} \ \forall n \geq k \ \ x_n = 0\}$ is a meager subset of ℓ_∞.

(12+12+12+12 pts) **2.** Consider the normed space $(\ell_1, \|\cdot\|_\infty)$. For each $n \in \mathbb{N}$, define the linear functional $S_n : \ell_1 \to \mathbb{R}$ by

$$S_n(\mathbf{a}) = \sum_{i=0}^{n} a_i$$

for all $\mathbf{a} \in \ell_1$, i.e. S_n takes each sequence in ℓ_1 to the n-th partial sum of the corresponding series. (**Warning.** Note that in this question the norm on ℓ_1 is $\|\cdot\|_\infty$ and not $\|\cdot\|_1$.)

a) Prove that S_n is bounded for every $n \in \mathbb{N}$.

b) Prove that, for all $\mathbf{a} \in \ell_1$, we have $\sup_{n\in\mathbb{N}} |S_n(\mathbf{a})| < \infty$.

c) Prove that $\sup_{n\in\mathbb{N}} \|S_n\| = \infty$.

d) Why do Part (b) and (c) together not contradict the uniform boundedness principle?

***(12+12+12 pts)* 3.**

a) State either the First Isomorphism Theorem or the Open Mapping Theorem.

For the remainder of this question, consider the Banach space $(\ell_1, \|\cdot\|_1)$ and its subspace

$$S = \left\{ \mathbf{a} \in \ell_1 : \sum_{n=0}^{\infty} a_n = 0 \right\}$$

b) Prove that $(\ell_1/S, \|\cdot\|_q)$ is isomorphic to $(\mathbb{R}, |\cdot|)$ where $\|\cdot\|_q$ is the quotient norm.

c) Let $Q \subseteq \ell_1$ be an algebraic complement of S in ℓ_1, that is, Q is a subspace such that $\ell_1 = S \bigoplus Q$. Prove that $\ell_1 = S \bigoplus_{top} Q$, that is, S and Q are topologically complementary subspaces.

(12 pts) **4.** Consider the quotient vector space ℓ_∞ / c_{00} and the map $\|\cdot\|_q : \ell_\infty / c_{00} \to \mathbb{R}$ given by

$$\|\mathbf{x} + c_{00}\|_q = \inf_{\mathbf{y} \in c_{00}} \|\mathbf{x} - \mathbf{y}\|_\infty$$

for all $\mathbf{x} \in \ell_\infty$. Show that $\|\cdot\|_q$ is **not** a norm by explicitly finding a non-zero element $\mathbf{a} + c_{00} \in \ell_\infty / c_{00}$ such that $\|\mathbf{a} + c_{00}\|_q = 0$.