PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION $120+\epsilon$ MINUTES
5 QUESTIONS ON 4 PAGES		TOTAL 120 (+Bonus 6) POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

Throughout this exam, we shall work with normed spaces over the field $\mathbb{K} = \mathbb{R}$. In your solutions, you can quote all facts that we have learned in class (or other classes such as MATH349) without proof. However, if you wish to use a (non-trivial) fact that is not covered in class, then you are supposed to prove it.

***(12+12 pts)* 1.** Consider the normed space $(\ell_\infty, \|\cdot\|_\infty)$.

a) Show that the map $p : \ell_\infty \to \mathbb{R}$ given by $p(\mathbf{a}) = \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n$ is a seminorm.

b) Prove that there exists a continuous linear functional $\Phi : \ell_\infty \to \mathbb{R}$ such that

$$|\Phi(\mathbf{a})| \leq \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n$$

$$\Phi(1,0,1,0,\dots) = \frac{1}{452}$$

Then find $\Phi(0,1,0,1,\dots)$.

***(12+12+12+12(+6) pts)* 2.** Consider the set

$$P[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is a polynomial function with coefficients in } \mathbb{R}\}$$

In other words, $P[0,1]$ is the subspace of $C[0,1]$ spanned by the functions $\{x^k : k \in \mathbb{N}\}$. Consider the norm $\|\cdot\|_{\mathrm{s}}$ given by

$$\|f\|_{\mathrm{s}} = \sum_{i=0}^{n} |a_i| \quad \text{where } f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for all $f \in P[0,1]$. For the rest of this question, the vector space $P[0,1]$ is endowed with the norm $\|\cdot\|_{\mathrm{s}}$ unless stated otherwise.

a) Prove that the normed space $(P[0,1], \|\cdot\|_{\mathrm{s}})$ is *not* Banach by exhibiting a Cauchy sequence which is not convergent.

Hint. Try to observe some "relationship" between $P[0,1]$ and c_{00} as vector spaces. Then recall how we proved that c_{00} is not Banach and try to modify that argument appropriately.

b) Prove that the integral norm $\|\cdot\|_1 : P[0,1] \to \mathbb{R}$ given by $\|f\|_1 = \displaystyle\int_0^1 |f(x)| dx$ is **not** equivalent to the norm $\|\cdot\|_{\mathrm{s}}$.

c) Prove that the linear map $D : P[0,1] \to P[0,1]$ given by $D(f) = f'$ is not bounded, where f' is the derivative of f.

d) Consider the map $T : P[0,1] \to \ell_\infty$ given by

$$T(f) = \left(f\left(\frac{1}{n+1}\right)\right)_{n\in\mathbb{N}} = \left(f(1), f\left(\frac{1}{2}\right), f\left(\frac{1}{3}\right), \dots\right)$$

Prove that T is a continuous linear map and find $\|T\|$.

Bonus (6 pts). Prove also that T is injective.

(12+12 pts) **3. a)** State the definition of a Schauder basis of an infinite-dimensional normed space $(E, \|\cdot\|)$.

b) What is an example of a normed space with a Schauder basis whose dual space does not have a Schauder basis? (Recall that you can quote any fact from class without proof.)

***(12 pts)* 4.** Let $(E, \|\cdot\|)$ be a normed space over $\mathbb{R}$. Prove that there exists an isometric isomorphism $T : L(\mathbb{R}, E) \to E$.

***(12 pts)* 5.** Let $(E, \|\cdot\|)$ be a normed space over $\mathbb{R}$ and let $\{x_n : n \in \mathbb{N}\} \subseteq E$ be a set of linearly independent vectors. Prove that there exists a set $\{f_n : n \in \mathbb{N}^+\} \subseteq E'$ of continuous linear functionals such that

$$f_n(x_m) < f_n(x_n) \text{ for all integers } 0 \leq m < n < \infty$$