PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 150 MINUTES
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

Throughout this exam, we shall work with normed spaces over the field $\mathbb{K} = \mathbb{R}$. In your solutions, you can quote all facts that we have learned in class (or other classes such as MATH349) without proof. However, if you wish to use a (non-trivial) fact that is not covered in class, then you are supposed to prove it.

***(10+10+10+10 pts)* 1.** For each $n \in \mathbb{N}^+$, set $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ to be the maps given by

$$f_n(x) = \sin(2\pi n x) \text{ and } g_n(x) = \cos(2\pi n x)$$

for all $x \in [0,1]$. Consider the vector subsubspace $E = \langle f_n, g_n : n \in \mathbb{N}^+ \rangle$ of $C([0,1])$. Recall that these functions are linearly independent and hence, given any $f \in E$, f can be uniquely written as a sum of these basis elements as

$$f = \sum_{n=1}^{\infty} \alpha_n f_n + \sum_{n=1}^{\infty} \beta_n g_n$$

where all but finitely many of α_n's and β_n's are zero. In this case, define

$$\|f\| = \sum_{n=1}^{\infty} |\alpha_n| + \sum_{n=1}^{\infty} |\beta_n|$$

a) Show that the map $\|\cdot\| : E \to \mathbb{R}$ is a norm.

b) Show that the closed unit ball $S = \{f \in E : \|f\| \leq 1\}$ is not compact but is σ-compact, that is, a countable union of compact sets.

c) Show that $(E, \|\cdot\|)$ is not a Banach space. (**Hint.** There is no need to search for non-convergent Cauchy sequences. Try to understand how much it takes to generate E.)

d) Determine whether or not the differentiation map $D : E \to E$ given by $D(f) = f'$ is a bounded map.

***(10+10+10 pts)* 2.** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces and $T \in L(E, F)$. We define the *adjoint* of T to be the bounded linear map $T^* : F' \to E'$ given by $T^*(\varphi) = \varphi \circ T$. In other words, we have

$$T^*(\varphi)(e) = (\varphi \circ T)(e) = \varphi(T(e))$$

for all $\varphi \in F'$ and $e \in E$.

a) Let $\varphi \in F'$ and consider the bounded linear functional $T^*(\varphi) : E \to \mathbb{R}$. Write down the definition of $\|T^*(\varphi)\|$.

b) Show that $\|T^*\| \leq \|T\|$. (By the Hahn-Banach theorem, we actually have the other direction as well. Thus $\|T^*\| = \|T\|$ but I want to you to prove only one direction.)

c) Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of elements of F' and $\varphi \in F'$. Show that if $\varphi_n \rightharpoonup^* \varphi$ in F', then $T^*(\varphi_n) \rightharpoonup^* T^*(\varphi)$ in E'.

***(10 pts)* 3.** State the Banach-Alaoğlu-Bourbaki theorem.

(10 pts) **4.** Consider the normed space $(\ell_\infty, \|\cdot\|_\infty)$ and the seminorm $p : \ell_\infty \to \mathbb{R}$ given by $p(\mathbf{a}) = \limsup_{n\to\infty} |a_n|$. Prove that there exists a continuous linear functional $\Phi : \ell_\infty \to \mathbb{R}$ such that

$$|\Phi(\mathbf{a})| \leq p(\mathbf{a}) = \limsup_{n\to\infty} |a_n| \quad \text{and} \quad \Phi(1,0,1,0,\dots) = \frac{1}{452} \quad \text{and} \quad \Phi(0,1,0,1,\dots) = 0.$$

(10 pts) **5.** Let $(E, \|\cdot\|)$ be an infinite-dimensional normed space and $U \subseteq E$ be a weakly-open set such that $\mathbf{0} \in U$. Show that there exists an infinite-dimensional norm-closed subspace C with $C \subseteq U$. (**Hint.** Pick an appropriate neighborhood basis element of $\mathbf{0}$ and try using the data encoded in that.)