MATH 406 2021-1: Take-Home Assignment

In this take-home homework assignment, you are going to learn some basic facts about the infinitary logic $\mathbf{L}_{\omega_1\omega}$.

Let us first give some motivation before we introduce $\mathbf{L}_{\omega_1\omega}$. Recall that our whole model-theoretic discussion is based on formalizing informal objects of logic as mathematical objects within some set-theoretic metatheory and arguing about these objects mathematically. While we have built and analyzed the model theory of the classical first-order *finitary* logic in this course, there is nothing preventing us from generalizing these ideas and constructing formalized *infinitary* logical systems. These infinitary logical systems clearly only exist as imaginary mathematical objects of our metatheory and are *not of actual practical use* in real life as a logical system. Nevertheless, they are useful for mathematical purposes. In what follows, as before, we will work in a set-theoretic metatheory that is capable of talking about cardinalities, such as ZFC.

The construction of $\mathbf{L}_{\omega_1\omega}$ is almost identical to the usual first-order logic. We fix a language $\mathcal{L}$ and define the terms and atomic formulas of $\mathcal{L}$ as usual. The only difference is that, while constructing formulas recursively, we change two rules. The set $\mathrm{Form}(\mathbf{L}_{\omega_1\omega})$ of formulas of $\mathbf{L}_{\omega_1\omega}$ is the smallest set satisfying the following properties.

- $\mathrm{Form}(\mathbf{L}_{\omega_1\omega})$ contains all atomic formulas.
- If $\varphi, \psi \in \mathrm{Form}(\mathbf{L}_{\omega_1\omega})$ and x is a variable symbol, then

$$\neg\varphi,\ \varphi \to \psi,\ \forall x\varphi,\ \exists x\varphi \in \mathrm{Form}(\mathbf{L}_{\omega_1\omega})$$

- If $\{\varphi_i\}_{i\in I} \subseteq \mathrm{Form}(\mathbf{L}_{\omega_1\omega})$ is a set of formulas with some *countable* index set I, then

$$\bigvee_{i\in I}\varphi_i,\ \bigwedge_{i\in I}\varphi_i \in \mathrm{Form}(\mathbf{L}_{\omega_1\omega})$$

In other words, the formulas of $\mathbf{L}_{\omega_1\omega}$ are obtained exactly in the same way as those of the usual first-order logic except: We allow *not only finite but also countably infinite disjunctions and conjunctions.* You should think of these infinitary disjunctions and conjunctions not as "strings that can be written down in real life" but as abstract formal mathematical objects, perhaps by comparing this situation with polynomials versus formal power series.

For example, the following is a well-formed formula of $\mathbf{L}_{\omega_1\omega}$ over the language $\mathcal{L} = \{\cdot, e\}$ of group theory.

$$\forall g \bigvee_{n\in\mathbb{N}^+} \varphi_n(g) \qquad : \qquad \forall g\ (g = e\ \vee\ g\cdot g = e\ \vee\ g\cdot g\cdot g = e\ \vee\ \dots)$$

where $\varphi_n(g) \quad : \quad g \dots g = e$ with g appearing n times on the left-hand side. This sentence simply says "every element has finite order" which we know we cannot express in the usual finitary first-order logic due to a compactness argument. Free variables and the set $\mathrm{Sent}(\mathbf{L}_{\omega_1\omega})$ of sentences of $\mathbf{L}_{\omega_1\omega}$ can similarly be defined as usual.

The semantics of $\mathbf{L}_{\omega_1\omega}$, more precisely, the satisfaction relation $\models$ for $\mathbf{L}_{\omega_1\omega}$ is defined in the exact same way as the satisfaction relation of the usual first-order logic in a recursive manner, except that we replace the recursive rules

- $\mathcal{A} \models \varphi \vee \psi\ [s]$ holds if $\mathcal{A} \models \varphi\ [s]$ holds or $\mathcal{A} \models \psi\ [s]$ holds,
- $\mathcal{A} \models \varphi \wedge \psi\ [s]$ holds if $\mathcal{A} \models \varphi\ [s]$ holds and $\mathcal{A} \models \psi\ [s]$ holds,

which we had for the usual first-order logic by the recursive rules

- $\mathcal{A} \models \bigvee_{i \in I} \varphi_i\ [s]$ holds if $\mathcal{A} \models \varphi_i\ [s]$ holds for some $i \in I$,
- $\mathcal{A} \models \bigwedge_{i \in I} \varphi_i\ [s]$ holds if $\mathcal{A} \models \varphi_i\ [s]$ holds for all $i \in I$.

due to the nature of the construction of formulas. I shall not go into the details of semantics which, at this point, you should be able to work out on your own. The terminology, for example, the use of the word "model", is generalized to $\mathbf{L}_{\omega_1\omega}$ in an expected manner.

What I have described above is indeed a special case of a more general construction of the infinitary logic $\mathbf{L}_{\kappa\lambda}$ where you allow conjunction and disjunction of less-than-κ many formulas and simultaneous quantification of less-than-λ many variable symbols. The classical first-order logic that we learned in class is simply $\mathbf{L}_{\omega\omega}$. Infinitary logics $\mathbf{L}_{\kappa\lambda}$ have been studied for a long time. The curious reader may consult to the books

- "Lectures on Infinitary Model Theory" by David Marker,
- "Model Theory for Infinitary Logic" by H. Jerome Keisler,
- "Large Infinitary Languages" by M. A. Dickmann

to learn more. For this take-home assignment, I do not believe that you will have to consult to the last one, but you may want to take a look the former two. Indeed, the purpose of this take-home assignment is to let you know that we have only seen the tip of the iceberg in class and there is a whole world of interesting ideas for you to explore. Thus you are suggested and indeed, expected to use outside sources such as books, articles, internet sources etc. whenever you feel that it is necessary.

Remarks about grading. You have 5 questions to answer. Each question is worth 3 points. Your score from this take-home assignment is

$$\min\{\text{the points you collected}, 10\}$$

Consequently, even if you collect more than 10 points in this assignment, you will have received 10 points at the end. You may think of this situation as having a "5-point buffer zone". Since this is a take-home exam and you have more than enough time, please write your solutions in an organized and detailed manner.

Question 1. Let $\mathcal{L} = \{\cdot, e, \cdot^{-1}\}$ be the language of group theory. Write down a sentence $\varphi \in \mathrm{Sent}(\mathbf{L}_{\omega_1\omega})$ such that, for any $\mathcal{L}$-structure $\mathcal{G}$, we have

$$\mathcal{G} \models \varphi \text{ iff } \mathcal{G} \text{ is a finitely generated group}$$

Briefly explain why the sentence φ that you propose actually works.

Hint. Consider the following basic case of 2-generated groups. Let G be a 2-generated group, say, $G = \langle a, b \rangle$. How many possible finite products of the elements a and b and their inverses are there? How are you supposed to say that every element is equal to one of these finite product in $\mathbf{L}_{\omega_1\omega}$? What would happen if you use an arbitrary $n \in \mathbb{N}^+$ instead of 2? How would you combine these ideas?

Question 2. Let $\mathcal{L} = \{+, \cdot, 0, S, <\}$. Recall that the true arithmetic

$$TA = Th_{\mathcal{L}}(\mathbb{N}, +, \cdot, 0, S, <)$$

is the set of all usual finitary first-order sentences that are true in the structure $(\mathbb{N}, +, \cdot, 0, S, <)$. As you may remember, we have built non-standard models of TA, that is, models of TA that are not isomorphic $(\mathbb{N}, +, \cdot, 0, S, <)$.

Find a sentence $\varphi \in \mathrm{Sent}(\mathbf{L}_{\omega_1\omega})$ such that every model of

$$TA \cup \{\varphi\}$$

is isomorphic to $(\mathbb{N}, +, \cdot, 0, S, <)$. For this question, you do not have to justify your answer, simply write down the sentence.

Hint. In a non-standard model of TA, we have these "non-standard numbers" that appear right after a copy of $\mathbb{N}$ which is an initial segment of this model. Try to right down a sentence which just prevents such elements from existing.

Question 3. Does the Compactness Theorem for the usual first-order logic generalize to the infinitary logic $\mathbf{L}_{\omega_1\omega}$? If yes, prove it. If no, provide a counterexample.

Question 4. Does the upward Löwenheim-Skolem theorem for the usual first-order logic generalize to the infinitary logic $\mathbf{L}_{\omega_1\omega}$? If yes, prove it. If no, provide a counterexample.

Question 5. Does the downward Löwenheim-Skolem theorem for the usual first-order logic generalize to the infinitary logic $\mathbf{L}_{\omega_1\omega}$? If yes, prove it. If no, provide a counterexample.