PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 150 MINUTES
1 QUESTION, 9 PARTS ON 4 PAGES		TOTAL 126 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(9 × 14=126 pts) Throughout this exam, we shall work in the language $\mathcal{L} = \{f\}$ where f is a unary function symbol. Consider the theory T which consists of the following two sentences.

$$\forall x \; \forall y \quad (f(x) = f(y) \; \rightarrow \; x = y)$$
$$\forall y \; \exists x \quad y = f(x)$$

In other words, T is the theory of bijections of a set.

a) How many models of T are there whose universe is the set $\{1, 2, \dots, 406\}$?

b) Find an $\mathcal{L}$-structure $\mathcal{M} = \left(M, f^{\mathcal{M}}\right)$ such that $|M| = 406$, $\mathcal{M} \models T$ and $\mathcal{M}$ does not have any proper substructures. (**Hint.** Remember that a substructure is necessarily closed under the application of the interpretation of the functions symbols. Find a suitable structure for which this procedure would exhaust all elements.)

c) Consider the $\mathcal{L}$-structure $\mathcal{A} = \left(\mathbb{Z}, f^{\mathcal{A}}\right)$ where the interpretation of the function symbol f is the map $f^{\mathcal{A}} : \mathbb{Z} \to \mathbb{Z}$ given by

$$f^{\mathcal{A}}(n) = \begin{cases} n+2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

Let $\mathcal{B} \subseteq \mathcal{A}$ be the substructure of $\mathcal{A}$ whose universe is the set of even integers, that is, $\mathcal{B} = \left(2\mathbb{Z}, f^{\mathcal{B}}\right)$ where $f^{\mathcal{B}} = f^{\mathcal{A}} \restriction 2\mathbb{Z}$. Show that $\mathcal{B}$ is not an elementary substructure of $\mathcal{A}$.

d) Consider the $\mathcal{L}$-structure $\mathcal{A} = \left(\mathbb{Z}, f^{\mathcal{A}}\right)$ defined in Part (c). Let $X \subseteq \mathbb{Z}$ be definable without parameters in the structure $\mathcal{A}$ such that $406 \in X$. Show that $2\mathbb{Z} \subseteq X$.
(**Hint.** What is the relationship between definability and automorphisms?)

For the rest of this question, consider the theory T' which is obtained by adding the following infinitely many sentences to T.

$$\forall x\ f(x) \neq x$$

$$\forall x\ f(f(x)) = x$$

$$\exists x_1\ \exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \qquad \text{for each integer } n \geq 2$$

In other words, T' is the theory of fixed-point-free involutions of an infinite set.

e) Show that T' is $\aleph_0$-categorical. (**Hint.** Pick an arbitrary countable model of T' and try to understand how the "orbits" of the interpretation of f partitions the model.)

f) Show that T' is complete.

g) Let $\mathcal{W} \models T'$ be a model of T'. Does there necessarily exist a countable elementary substructure $\widetilde{\mathcal{V}} \preccurlyeq \mathcal{W}$? Why?

h) Let $\mathcal{W} \models T'$ be as in Part (g) and let $\mathcal{V} \models T'$ be a countable model of T'. Show that there exists an elementary embedding $\pi : V \to W$.
(**Hint.** Recall what you had in parts (e) and (g).)

For each $n \in \mathbb{N}^+$, set $C_n = \{1, 2, \ldots, 2n\}$ and consider the $\mathcal{L}$-structure $\mathcal{C}_n = (C_n, f^{\mathcal{C}_n})$ where the interpretation $f^{\mathcal{C}_n} : C_n \to C_n$ of the function symbol f is the permutation

$$(1\ 2) \ldots (2n-1\ 2n)$$

which is given in the usual cycle notation.

i) Let $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N}^+)$ be a non-principal ultrafilter on $\mathbb{N}^+$ and consider the ultraproduct

$$\mathcal{X} = \left(X, f^{\mathcal{X}}\right) = \prod_{n \in \mathbb{N}^+} \mathcal{C}_n \ / \ \mathcal{U}$$

Show that $\mathcal{X} \models T'$.