PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 150 MINUTES
5 QUESTION ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(10+10=20 pts) **1. a)** Find a suitable language $\mathcal{L}$ and an $\mathcal{L}$-theory $T \subseteq Sent_{\mathcal{L}}$ such that the class of models of T is precisely the class of pairs of the form (A, R) where A is a non-empty set and R is an equivalence relation on A.

b) Find a sentence $\varphi \in Sent_{\mathcal{L}}$ such that the theory $T' = T \cup \{\varphi\}$ is κ-categorical for all non-zero cardinals κ. Explain your reasoning in detail.

(10+10+10=30 pts) **2.** For this question, fix an arbitrary language $\mathcal{L}$. In what follows, all $\mathcal{L}$-theories are assumed to be consistent.

a) Prove that for any $\mathcal{L}$-theory T, there exists a complete $\mathcal{L}$-theory T' such that $T \subseteq T'$.

b) Let T be a complete $\mathcal{L}$-theory. Prove that any two models of T are elementarily equivalent.

c) Let T be an $\mathcal{L}$-theory. Suppose that there exists a model $\mathcal{A} \models T$ such that for every model $\mathcal{B} \models T$ there is an elementary embedding $\pi : A \to B$. Show that T is complete.

(10+10=20 pts) **3.** In this question, we shall work in the language $\mathcal{L} = \{<\}$ that consists of a single binary relation symbol. For each $n \in \mathbb{N}^+$, consider the $\mathcal{L}$-structure

$$\mathcal{A}_n = \left(\{1, 2, \ldots, n\}, <^{\mathcal{A}_n}\right)$$

where the interpretation $<^{\mathcal{A}_n}$ of the symbol $<$ is the usual less-than relation on integers. Observe that each $\mathcal{A}_n$ is a strictly linearly ordered set. Fix some non-principal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N}^+)$ on $\mathbb{N}^+$ and consider the ultraproduct

$$\mathcal{X} = \left(X, <^{\mathcal{X}}\right) = \prod_{n \in \mathbb{N}^+} \mathcal{A}_n \;/\; \mathcal{U}$$

Recall that, by Łoś's theorem, $\mathcal{X} = \left(X, <^{\mathcal{X}}\right)$ is a strictly linearly ordered set as well.

a) If it exists, find the greatest element of $\mathcal{X}$; if it does not, write "no greatest element". For this part of this question only, you do not have to justify your answer.

b) Consider the formula $\psi(x, y) : (x < y \wedge \neg\exists z(x < z \wedge z < y))$. Given $\mathbf{a}, \mathbf{b} \in \mathcal{X}$, we say that $\mathbf{b}$ is the immediate successor of $\mathbf{a}$ if we have $\mathcal{X} \models \psi(\mathbf{a}, \mathbf{b})$. Given that it exists, find the immediate successor of the element

$$\overline{(1, 1, 1, \ldots)} \in X$$

and explain in detail why the element you propose is the immediate successor.

(10 pts) **4.** Consider the language $\mathcal{L} = \{<\}$ that consists of a single binary relation symbol. **Using the compactness theorem**, show that there does **not** exist an $\mathcal{L}$-theory $\Sigma \subseteq Sent_{\mathcal{L}}$ such that for any $\mathcal{L}$-structure $\mathcal{A} = (A, <^{\mathcal{A}})$, we have $\mathcal{A} \models \Sigma$ if and only if $\mathcal{A}$ is a strictly well-ordered set.

(10+10=20 pts) **5. a)** State the Omitting Types Theorem.

b) Let $\mathcal{L}$ be a countable language and T be a complete $\aleph_0$-categorical $\mathcal{L}$-theory. Show that every complete n-type of T is principal. (**Hint.** Recall that any n-type of a complete theory over a countable language can be realized in some countable model.)