PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 90+ϵ MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 80 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(10+15+15 pts) **1. (a)** Let (X, d) be a metric space. Complete the following definition appropriately: A set $S \subseteq X$ is said to be compact if for every collection $\{O_\alpha\}_{\alpha \in I}$ of open sets with $S \subseteq \bigcup_{\alpha \in I} O_\alpha$ .... there exists a finite set $F \subseteq I$ such that $S \subseteq \bigcup_{\alpha \in F} O_\alpha$

(b) Let (X, d) be a compact metric space and $\{C_\alpha\}_{\alpha \in I}$ be a collection of closed subsets of X. Suppose that the collection $\{C_\alpha\}_{\alpha \in I}$ has the finite intersection property, that is, for any finite $F \subseteq I$ we have $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$. Prove that $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$.

Assume towards a contradiction that $\bigcap_{\alpha \in I} C_\alpha = \emptyset$. Then, by De Morgan's law, we obtain $\bigcup_{\alpha \in I} X \setminus C_\alpha = X$ and hence $\{X \setminus C_\alpha\}_{\alpha \in I}$ is an open cover for X. Since X is compact, there exists a finite $F \subseteq I$ such that $\bigcup_{\alpha \in F} X \setminus C_\alpha = X$. It now follows from De Morgan's law that $\bigcap_{\alpha \in F} C_\alpha = \emptyset$ which contradicts the given assumption.

(c) Let (X, d) be a metric space, $K \subseteq X$ be a compact subset and $w \in X$. Show that $\inf_{x \in K} d(w, x) = d(w, k)$ for some $k \in K$.

Hint. Consider the map $x \mapsto d(w, x)$ from K to $\mathbb{R}$. First show that this map is continuous. What do you know about continuous maps on compact spaces?

We first show that the map $x \mapsto d(w, x)$ from K to $\mathbb{R}$ is continuous, and indeed, uniformly continuous. Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in K$. Observe that, by the triangle inequality and the symmetry of d, we have $d(w, y) - d(x, y) \leq d(w, x) \leq d(w, y) + d(x, y)$ and hence $|d(w, x) - d(w, y)| \leq d(x, y)$. Therefore,

$$\text{if } d(x, y) < \delta, \text{ then } |d(w, x) - d(w, y)| \leq d(x, y) < \delta = \epsilon.$$

This shows that this map is uniformly continuous and in particular, continuous.

Since the map $x \mapsto d(w, x)$ is continuous on K and K is compact, by the Extreme Value theorem, the map $x \mapsto d(w, x)$ attains its minimum value on K. More precisely, there exists $k \in K$ such that $d(w, k) = \inf_{x \in K} d(w, x)$.

***(10+10 pts)* 2.** Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = e^{\frac{x^2}{n+1}}$ for each $n \in \mathbb{N}$ and let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = 1$.

(a) Show that $f_n \longrightarrow f$ uniformly.

Recall that $f_n \longrightarrow f$ uniformly iff $f_n \longrightarrow f$ with respect to the sup metric. Observe that, since f_n is increasing and $f_n(x) \geq 1$ on $[0,1]$, we have that

$$0 \leq d_{\text{sup}}(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| \leq \sup_{x\in[0,1]} \left| e^{\frac{1}{n+1}} - 1 \right| = e^{\frac{1}{n+1}} - 1 \xrightarrow{n\to\infty} 0$$

and hence $\lim_{n\to\infty} d_{\text{sup}}(f_n, f) = 0$. This implies that $f_n \longrightarrow f$ with respect to the sup metric and so $f_n \longrightarrow f$ uniformly.

(b) Compute $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)dx$. You can freely use the fact in Part (a) even if you could not prove it.

Since $f_n \longrightarrow f$ uniformly by Part (a), we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 f(x)dx = \int_0^1 1\ dx = 1$$

***(10+10 pts)* 3. (a)** Let (X,d) be a metric space. State the definition of a totally bounded (or sometimes also called, precompact) subset of X: A set $S \subseteq X$ is said to be totally bounded if ....for every $\epsilon > 0$ there exists $x_1, x_2, \ldots, x_n \in X$ such that $S \subseteq \bigcup_{i=1}^n B_d(x_i, \epsilon)$.

(d) Consider the metric space $(\mathbb{Z}^+, d)$ where $d(k,\ell) = \left|\dfrac{1}{k} - \dfrac{1}{\ell}\right|$. Show that $\mathbb{Z}^+$ is totally bounded.

Let $\epsilon > 0$. By the Archimedean property, we can find $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \epsilon$. We claim that the points $1, 2, \ldots, n$ satisfies the required property in the definition of totally boundedness. Observe that, for any $k \geq n$, we have

$$d(n,k) = \left|\frac{1}{n} - \frac{1}{k}\right| = \frac{1}{n} < \epsilon$$

and hence $k \in B_d(n,\epsilon)$. Thus we have $\{n, n+1, n+2, \ldots\} \subseteq B_d(n,\epsilon)$. Since $i \in B_d(i,\epsilon)$ for each $i = 1, \ldots, n$ as well, it now follows that $\mathbb{Z}^+ \subseteq \bigcup_{i=1}^n B_d(i,\epsilon)$. Therefore $\mathbb{Z}^+$ is totally bounded.