PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 90+ϵ MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 80 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

***(10+14 pts)* 1.** **(a)** Let (X, d) and $(Y, \overline{d})$ be metric spaces and $f : X \to Y$ be a map. State the definition of uniform continuity: The map f is uniformly continuous iff ...

For every $\epsilon \in \mathbb{R}^+$ there exists $\delta \in \mathbb{R}^+$ such that for all $a, b \in X$ if $d(a, b) < \delta$ then $\overline{d}(f(a), f(b)) < \epsilon$.

(b) Consider the metric space $(2^{\mathbb{N}}, d)$ where $2^{\mathbb{N}} = \{(a_n)_{n\in\mathbb{N}} : \ a_n \in \{0, 1\} \text{ for all } n \in \mathbb{N}\}$ and $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to [0, \infty)$ is given by

$$d\left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right) = \begin{cases} 0 & \text{if } (a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}} \\ \dfrac{1}{\min\{k \in \mathbb{N} : a_k \neq b_k\} + 1} & \text{if } (a_n)_{n\in\mathbb{N}} \neq (b_n)_{n\in\mathbb{N}} \end{cases}$$

Show that the map $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by

$$\varphi\big((a_0, a_1, a_2, \dots)\big) = (a_0, a_2, a_4, \dots) \text{ for all } (a_n)_{n\in\mathbb{N}} \in 2^{\mathbb{N}}$$

is uniformly continuous.

We first make the following observation. For any $\mathbf{a}, \mathbf{b} \in 2^{\mathbb{N}}$, if the first k terms of $\mathbf{a}$ and $\mathbf{b}$ are the same, then the first $\lceil k/2 \rceil$ terms of $\varphi(\mathbf{a})$ and $\varphi(\mathbf{b})$ are the same by the definition of φ, where $\lceil \cdot \rceil$ denotes the ceiling function. We shall now prove that φ is uniformly continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon/2$. Let $\mathbf{a}, \mathbf{b} \in 2^{\mathbb{N}}$. Suppose that $d(\mathbf{a}, \mathbf{b}) < \delta$. By definition, we know that $d(\mathbf{a}, \mathbf{b}) = \frac{1}{k+1}$ for some $k \in \mathbb{N}$ and moreover, this k satisfies that $a_i = b_i$ for all $0 \leq i < k$, that is, the first k terms of $\mathbf{a}, \mathbf{b}$ are the same. By the observation, we know that the first $\lceil k/2 \rceil$ terms of $\varphi(\mathbf{a})$ and $\varphi(\mathbf{b})$ are the same and, using that $\frac{1}{k+1} < \delta$, we obtain

$$d(\varphi(\mathbf{a}), \varphi(\mathbf{b})) \leq \frac{1}{\lceil k/2 \rceil + 1} < \frac{1}{k/2 + 1} = \frac{2}{k+2} < \frac{2}{k+1} < 2\delta = \epsilon$$

Hence φ is uniformly continuous.

(14+14+14 pts) 2. Consider the metric space $(\mathbb{Z}^+, d)$ where $d(k, \ell) = \left|\frac{1}{k} - \frac{1}{\ell}\right|$. You are **given** that $B_d\left(k, \frac{1}{2k^2}\right) = \{k\}$.

(a) Prove that any convergent sequence in $(\mathbb{Z}^+, d)$ is eventually constant, i.e its terms are the same after some term.

Let $(k_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(\mathbb{Z}^+, d)$ with limit $k \in \mathbb{Z}^+$. Using the ϵ-N definition of convergence, we know that for $\epsilon = \frac{1}{2k^2}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(k_n, k) < \frac{1}{2k^2}$, equivalently, $k_n \in B_d\left(k, \frac{1}{2k^2}\right)$. By the given assumption that $B_d\left(k, \frac{1}{2k^2}\right) = \{k\}$, we now obtain that for all $n \geq N$ we have $k_n = k$. Hence $(k_n)_{n\in\mathbb{N}}$ is eventually constant.

(b) Prove that there exists a Cauchy sequence $(k_n)_{n\in\mathbb{N}}$ in $(\mathbb{Z}^+, d)$ that is **not** eventually constant.

Consider the sequence $(k_n)_{n\in\mathbb{N}}$ given by $k_n = n + 1$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Consequently, for all $m, n \geq N$ we have that

$$d(k_m, k_n) = \left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{\min\{m, n\} + 1} < \frac{1}{N} < \epsilon$$

Hence $(k_n)_{n\in\mathbb{N}}$ is Cauchy.

(c) Prove that $(\mathbb{Z}^+, d)$ is not complete. You can freely use the facts that you are supposed to prove in Part (a) and Part (b) even if you could not prove these.

It follows from Part (a) that the sequence in Part (b) cannot be a convergent sequence as it is not eventually constant. Therefore, there exists a Cauchy sequence in $(\mathbb{Z}^+, d)$ that is not convergent and hence $(\mathbb{Z}^+, d)$ is not complete.

(14 pts) 3. Consider the metric $\rho_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ on $\mathbb{R}^2$ given by

$$\rho_\infty\big((x, y), (x', y')\big) = \max\{|x - x'|, d_{\text{discrete}}(y, y')\}$$

where d_{discrete} is the discrete metric on $\mathbb{R}$. In other words, the metric space $(\mathbb{R}^2, \rho_\infty)$ is the product of the metric spaces $(\mathbb{R}, d_{\text{Euclidean}})$ and $(\mathbb{R}, d_{\text{discrete}})$. Find $(\alpha, \beta) \in \mathbb{R}^2$ and $\epsilon \in \mathbb{R}^+$ such that

$$B_{\rho_\infty}((\alpha, \beta), \epsilon) = \{(x, y) \in \mathbb{R}^2 : 1 < x < 349\}$$

Make sure to include your arguments for subset inclusions in both directions.

Choose $(\alpha, \beta) = (175, 0)$ and $\epsilon = 174$. We claim that the required equality holds.

$(\subseteq)$: Let $(x, y) \in B_{\rho_\infty}((175, 0), 174)$. Then, by the definition we know that

$$|x - 175| \leq \rho_\infty\big((x, y), (175, 0)\big) = \max\{|x - 175|, d_{\text{discrete}}(y, 0)\} < 174$$

It now follows from $|x - 175| < 174$ that $1 < x < 349$. So (x, y) is included on the right hand side.

$(\supseteq)$: Let $(x, y) \in \mathbb{R}^2$ be such that $1 < x < 349$. Then we have $-174 < x - 175 < 174$ and so $|x - 175| < 174$. Moreover, by the definition of discrete metric, we have that $d_{\text{discrete}}(y, 0) \leq 1 < 174$. Consequently, we have

$$\rho_\infty\big((x, y), (175, 0)\big) = \max\{|x - 175|, d_{\text{discrete}}(y, 0)\} < 174$$

Hence $(x, y) \in B_{\rho_\infty}((175, 0), 174)$. This completes the proof.