METU Department of Mathematics, Math 349, Midterm I, October 30, 2023

PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 90+ϵ MINUTES
2 QUESTIONS ON 2 PAGES		TOTAL 80 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(6+13+6+13 pts) **1.** **(a)** State the definition of an accumulation point of a subset. A point $x \in \mathbb{R}$ is said to be an accumulation point of $S \subseteq \mathbb{R}$ iff ....

For every $\epsilon \in \mathbb{R}^+$, the interval $(x-\epsilon, x+\epsilon)$ contains infinitely many elements of S. Equivalently, for every $\epsilon \in \mathbb{R}^+$, there exists $s \in S$ such that $s \in (x-\epsilon, x+\epsilon) \setminus \{x\}$.

(b) Let $S \subseteq \mathbb{R}$ be such that S is bounded and $\sup(S) \notin S$. Prove that $\sup(S)$ is an accumulation point of S.

Let $\epsilon \in \mathbb{R}^+$. Since $\sup(S) - \epsilon < \sup(S)$, by the definition of supremum, there exists $s \in S$ such that $\sup(S) - \epsilon < s \leq \sup(S)$. On the other hand, since $\sup(S) \notin S$, we must have that $\sup(S) - \epsilon < s < \sup(S)$. Therefore $s \in (\sup(S) - \epsilon, \sup(S) + \epsilon) \setminus \{\sup(S)\}$. It follows from the equivalent characterization of an accumulation point of a set that $\sup(S)$ is an accumulation point of S.

Note: If you use the former definition of an accumulation point, you can still solve this question with a similar idea by choosing $s_n \in S$ with $\sup(S) - \frac{\epsilon}{n} < s_n < \sup(S)$ for each $n \in \mathbb{N}^+$, however, the solution becomes longer.

(c) State the Bolzano-Weierstrass theorem.

Any bounded infinite subset of $\mathbb{R}$ has an accumulation point.

(d) Let $S \subseteq \mathbb{R}$ be a bounded subset with a *unique* accumulation point $\ell \in \mathbb{R} \setminus S$. Prove that S is countably infinite.

Hint. Ask yourself the following: If I were to take an arbitrary ball $(\ell - \frac{1}{n}, \ell + \frac{1}{n})$, how many elements of S could possibly live outside this ball? Can you obtain S from these?

Observe that, since $\ell \notin S$, we have that

$$S = \{\ell\}^c \cap S = \left(\bigcap_{n=1}^{\infty} \left(\ell - \frac{1}{n}, \ell + \frac{1}{n} \right) \right)^c \cap S = \bigcup_{n=1}^{\infty} \left(\left(\ell - \frac{1}{n}, \ell + \frac{1}{n} \right)^c \cap S \right)$$

For each $n \in \mathbb{N}^+$, the set $S_n = \left(\ell - \frac{1}{n}, \ell + \frac{1}{n}\right)^c \cap S$ must be finite since otherwise, being a bounded infinite subset of $\mathbb{R}$, by the Bolzano-Weierstrass theorem, the set S_n would have an accumulation point $\hat{\ell} \in S_n' \subseteq S'$; however, as closed sets contain all their accumulation points, we would have

$$\hat{\ell} \in S_n' \subseteq \left(\mathbb{R} \setminus \left(\ell - \frac{1}{n}, \ell + \frac{1}{n} \right) \right)' \subseteq \left(\mathbb{R} \setminus \left(\ell - \frac{1}{n}, \ell + \frac{1}{n} \right) \right) \subseteq \mathbb{R} \setminus \{\ell\}$$

and so $\ell \neq \hat{\ell}$, contradicting the existence of a unique accumulation point of S. But then S is a countable union of finite sets and hence is countable. Since finite subsets of $\mathbb{R}$ cannot have any accumulation points, S must also be infinite. Thus S is countably infinite.

(14+14+14 pts) 2. Consider the map $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to [0, \infty)$ given by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ \dfrac{1}{\max\{k \in \mathbb{N} :\ 2^k|m \text{ and } 2^k|n\} + 1} & \text{if } m \neq n \end{cases}$$

For example, $d(16, 28) = \frac{1}{2+1} = \frac{1}{3}$, $d(3, 32) = \frac{1}{0+1} = 1$ and $d(128, 256) = \frac{1}{7+1} = \frac{1}{8}$.

(a) Prove that d is a metric on $\mathbb{Z}^+$.

- Let $m, n \in \mathbb{Z}^+$. If $m = n$, then $d(m, n) = 0$ by definition. If $d(m, n) = 0$, then, since the value of d in the case $m \neq n$ is always positive, we must have that $m = n$.
- Let $m, n \in \mathbb{Z}^+$. If $m = n$, then $d(m, n) = 0 = d(n, m)$. If $m \neq n$, then

$$d(m,n) = \frac{1}{\max\{k \in \mathbb{N} :\ 2^k|m \text{ and } 2^k|n\} + 1} = \frac{1}{\max\{k \in \mathbb{N} :\ 2^k|n \text{ and } 2^k|m\} + 1} = d(n,m)$$

- Let $m, n, p \in \mathbb{Z}^+$. The triangle inequality $d(m, n) \leq (m, p) + d(p, n)$ clearly holds if any two of m, n, p are the same. So suppose that m, n, p are all distinct. Set $K = \max\{k \in \mathbb{N} :\ 2^k|m \text{ and } 2^k|n\}$ and $L = \max\{k \in \mathbb{N} :\ 2^k|p \text{ and } 2^k|n\}$. We split into cases
  - Case I ($K \geq L$): Then $d(m, n) = \frac{1}{K+1} \leq \frac{1}{L+1} = d(p, n) \leq d(m, p) + d(p, n)$.
  - Case II ($L > K$): Then $L \geq K+1$ and hence $2^{K+1}|p$ and $2^{K+1}|n$ by the definition of L. But then we must have $2^{K+1} \nmid m$ by the definition of K. But then we necessarily have that $\max\{k \in \mathbb{N} :\ 2^k|m \text{ and } 2^k|p\} \leq K$ which implies

$$d(m,n) = \frac{1}{K+1} \leq \frac{1}{\max\{k \in \mathbb{N} :\ 2^k|m \text{ and } 2^k|p\} + 1} = d(m,p) \leq d(m,p) + d(p,n)$$

  So d satisfies the triangle inequality and hence is a metric.

(b) Prove that $\{n\}$ is open with respect to d for every $n \in \mathbb{Z}^+$.
Let $n \in \mathbb{Z}^+$. We claim that $B_d(n, \frac{1}{n+1}) = \{n\}$. Assume towards a contradiction that there exists $m \in B_d(n, \frac{1}{n+1})$ with $m \neq n$. Then, since

$$0 < d(m,n) = \frac{1}{\max\{k \in \mathbb{N} :\ 2^k|m \text{ and } 2^k|n\} + 1} < \frac{1}{n+1}$$

we must have that $2^n|n$ which is a contradiction. Thus $B_d(n, \frac{1}{n+1}) = \{n\}$ and consequently $\{n\}$ is open as open balls are open.

Interlude. Let $\overline{d}$ be the discrete metric on $\mathbb{Z}^+$. It follows from Part (b) that every subset of $\mathbb{Z}^+$ is open with respect to d and hence, d and $\overline{d}$ are topologically equivalent metrics.

(c) Prove that d and $\overline{d}$ are not (strongly) equivalent metrics.
Assume towards a contradiction that d and $\overline{d}$ are (strongly) equivalent. Then there exist constants $c, C > 0$ such that for all $m, n \in \mathbb{Z}^+$ we have $c \cdot d(m, n) \leq \overline{d}(m, n) \leq C \cdot d(m, n)$. Set $N = \lfloor C \rfloor$. Then we have

$$1 = \overline{d}(2^N, 2^{N+1}) \leq C \cdot d(2^N, 2^{N+1}) \leq C \cdot \frac{1}{N+1} < C \cdot \frac{1}{C} = 1$$

which is a contradiction. Therefore d and $\overline{d}$ are not (strongly) equivalent