PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 120 MINUTES
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

(10+10+10+10 pts) **1.** **(a)** Let (X, d) be a metric space. State the definition of a nowhere dense subset of X, or, the definition of a contraction mapping on X.

Alternative I. A subset $S \subseteq X$ is called nowhere dense if $Int(\overline{S}) = \emptyset$.
Alternative II. A map $f : X \to X$ is called a contraction mapping if there exists $\alpha \in [0, 1)$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$.

(b) State the Banach fixed point theorem, or, the Baire category theorem.

Alternative I. (The Banach fixed point theorem) Let (X, d) be a complete metric space and $\varphi : X \to X$ be a contraction mapping. Then φ has a unique fixed point.
Alternative II. (The Baire category theorem) Let (X, d) be a complete metric space. Then, for any collection $\{O_n\}_{n\in\mathbb{N}}$ of open dense sets, $\bigcap_{n\in\mathbb{N}} O_n$ is dense.

(c) Recall that a set $S \subseteq \mathbb{R}$ is called G_δ if it is the countable intersection of open sets, that is, there exists a collection $\{O_n\}_{n\in\mathbb{N}}$ of open subsets of $\mathbb{R}$ such that $S = \bigcap_{n\in\mathbb{N}} O_n$. Show that $\mathbb{Q}$ is *not* G_δ.

Suppose towards a contradiction that $\mathbb{Q}$ is G_δ, say, $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} O_n$ for some countable collection $\{O_n\}_{n\in\mathbb{N}}$ of open subsets of $\mathbb{R}$. Observe that, for all $n \in \mathbb{N}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} \subseteq O_n$, we have that O_n is dense in $\mathbb{R}$. Moreover, for each $q \in \mathbb{Q}$, the set $\mathbb{R} \setminus \{q\}$ is an open dense subset of $\mathbb{R}$. It follows that the collection

$$\{O_n\}_{n\in\mathbb{N}} \cup \{\mathbb{R} \setminus \{q\}\}_{q\in\mathbb{Q}}$$

is a countable collection of open dense subsets of $\mathbb{R}$. But then, since $(\mathbb{R}, d_{\text{Euc}})$ is complete, by the Baire category theorem, the intersection

$$\bigcap_{n\in\mathbb{N}} O_n \cap \bigcap_{q\in\mathbb{Q}} (\mathbb{R} \setminus \{q\}) = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$$

must be dense, which is a contradiction.

(d) Let $X = \mathbb{Q} \cap [0,1]$ and consider the metric space (X, d_{Euc}). Find a contraction mapping $f : X \to X$ that does *not* have a fixed point.

Consider the map $f : X \to X$ given by $f(x) = \frac{1-x^2}{4}$. We first show that f is a contraction. Let $x, y \in X$. Then

$$|f(x) - f(y)| = \left|\frac{(1-x^2)-(1-y^2)}{4}\right| = \left|\frac{y^2-x^2}{4}\right| = \frac{|x-y||x+y|}{4} \leq \frac{|x-y|}{4} \cdot 2 \leq \frac{|x-y|}{2}$$

Therefore f is a contraction mapping with constant $\alpha = \frac{1}{2}$. If there were a fixed point $q \in X$ of the contraction f, then $f(q) = q$ would imply $1 - q^2 = 4q$ and, by the quadratic formula, we would have $q = \frac{-4 \pm \sqrt{20}}{2} = -2 \pm \sqrt{5}$, which is a contradiction as $q \in \mathbb{Q}$. Therefore f does not have any fixed points.

***(5+10 pts)* 2.** Let (X, d) be a metric space and $w \in X$.

(a) Show that the map $f : X \to \mathbb{R}$ given by $f(x) = d(x, w)$ is uniformly continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Observe that, by the triangle inequality and the symmetry of d, we have $d(w,y) - d(x,y) \leq d(w,x) \leq d(w,y) + d(x,y)$ and hence $|d(w,x) - d(w,y)| \leq d(x,y)$. Therefore,

$$\text{if } d(x,y) < \delta, \text{ then } |f(x) - f(y)| = |d(w,x) - d(w,y)| \leq d(x,y) < \delta = \epsilon.$$

This means that f is uniformly continuous.

(b) Suppose that (X, d) is connected and has at least two elements. Show that X must be uncountable.

Since X has at least two elements, there exists a point $a \in X$ different than $w \in X$. Set $r = d(a, w)$. Observe that X is connected, f is continuous, $f(a) = r > 0$ and $f(w) = 0$. It then follows from the Intermediate Value Theorem that $[0, r] \subseteq im(f)$. But then X must be uncountable because $|X| \geq |im(f)| \geq |[0, r]| = |\mathbb{R}|$.

<u>***(10+10+5 pts)* 3.**</u> Consider the metric space $(\mathbb{Z}^+, d)$ where $d(k,\ell) = \left|\frac{1}{k} - \frac{1}{\ell}\right|$.

(a) Show that $(\mathbb{Z}^+, d)$ is not complete.

THIS QUESTION WAS ASKED AND SOLVED IN MIDTERM 2.

(b) Show that $(\mathbb{Z}^+, d)$ is totally bounded.

THIS QUESTION WAS ASKED AND SOLVED IN MIDTERM 3.

(c) Determine whether or not $(\mathbb{Z}^+, d)$ is compact.

Recall that a metric space is compact iff it is complete and totally bounded. Since $(\mathbb{Z}^+, d)$ is not complete, it cannot be compact.

(10 pts) **4.** Let (X, d) be a compact metric space and $\{C_\alpha\}_{\alpha \in I}$ be a collection of closed subsets of X. Suppose that the collection $\{C_\alpha\}_{\alpha \in I}$ has the finite intersection property, that is, for any finite $F \subseteq I$ we have $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$. Prove that $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$.

THIS QUESTION WAS ASKED AND SOLVED IN MIDTERM 3.

(10 pts) **5.** Let $S \subseteq \mathbb{R}$ be a bounded subset with a *unique* accumulation point $\ell \in \mathbb{R} \setminus S$. Prove that S is countably infinite.

THIS QUESTION WAS ASKED AND SOLVED IN MIDTERM 1.