METU Department of Mathematics, Math 320, Midterm II, December 25, 2024, 17:40

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	6 questions on 4 pages, 120 minutes 100 points in total Justify your answer

1. (5+5+10 pts) a) State the definition of an ordinal number.

A set α is an ordinal if α is transitive and $(\alpha, \in_\alpha)$ is a strictly well-ordered set.

b) Complete the following statement of the principle of transfinite induction: A property $\varphi(x)$ of sets holds for all ordinal numbers if

- $\varphi(0)$ holds.
- $\varphi(S(\alpha))$ holds whenever $\varphi(\alpha)$ holds, for all ordinals α.
- For all limit ordinals γ, if $\varphi(\alpha)$ holds for all $\alpha < \gamma$, then $\varphi(\gamma)$ holds.

c) Recall that the addition $+$ and the multiplication $\cdot$ on ordinal numbers are recursively defined as follows.

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + S(\beta) &= S(\alpha+\beta) \\ \alpha + \gamma &= \sup\{\alpha+\theta : \theta \in \gamma\} \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot S(\beta) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \gamma &= \sup\{\alpha \cdot \theta : \theta \in \gamma\} \end{aligned}$$

for all ordinals α, β and limit ordinals γ. Recall also that the exponentiation with positive base on ordinals are recursively defined as follows.

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{S(\beta)} &= \alpha^\beta \cdot \alpha \\ \alpha^\gamma &= \sup\{\alpha^\theta : \theta \in \gamma\} \end{aligned}$$

for all ordinals $\alpha \geq 1$, β and limit ordinals γ. You are given that $\alpha^{\sup(X)} = \sup\{\alpha^\theta : \theta \in X\}$ and $\alpha \cdot \sup(X) = \{\alpha \cdot \theta : \theta \in X\}$ for all ordinals $\alpha \geq 1$ and for all non-empty set X of ordinals. Moreover, you are given that $0 + \alpha = \alpha$ for all ordinals α and that $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ for all ordinals α, β, γ and that $\alpha^\beta \geq 1$ for all ordinals $\alpha \geq 1$ and β.

Prove that, for all ordinals $\alpha \geq 1$ and for all ordinals β, γ, we have that

$$\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$$

WARNING: If you use an identity involving ordinal arithmetic other than the identities given in the question, you are supposed to prove it.

We shall prove this by transfinite induction on γ. Let $\alpha \geq 1$ and β.

- **Base case.** For $\gamma = 0$, by the definition of operations and given identities, we have
$$\alpha^{\beta+0} = \alpha^\beta = 0 + \alpha^\beta = \alpha^\beta \cdot 0 + \alpha^\beta = \alpha^\beta \cdot S(0) = \alpha^\beta \cdot 1 = \alpha^\beta \cdot \alpha^0$$
Hence the given identity holds for $\gamma = 0$.

- **Successor stage** Let γ be an ordinal. Suppose that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. Then, by the definition of operations, inductive assumption and given identities, we have
$$\alpha^{\beta+S(\gamma)} = \alpha^{S(\beta+\gamma)} = \alpha^{\beta+\gamma} \cdot \alpha = \left(\alpha^\beta \cdot \alpha^\gamma\right) \cdot \alpha = \alpha^\beta \cdot (\alpha^\gamma \cdot \alpha) = \alpha^\beta \cdot \alpha^{S(\gamma)}$$
Hence the given identity holds for $S(\gamma)$.

- **Limit case** Let γ be a limit ordinal. Suppose that $\alpha^{\beta+\theta} = \alpha^\beta \cdot \alpha^\theta$ holds for all $\theta < \gamma$. Then, by the definition of operations, inductive assumption and given identities, we have
$$\alpha^{\beta+\gamma} = \alpha^{\sup\{\beta+\theta :\, \theta<\gamma\}} = \sup\{\alpha^{\beta+\theta} : \theta < \gamma\} = \sup\{\alpha^\beta \cdot \alpha^\theta : \theta < \gamma\} = \alpha^\beta \cdot \sup\{\alpha^\theta : \theta < \gamma\} = \alpha^\beta \cdot \alpha^\gamma$$
Hence the given identity holds for γ.

It now follows from the principle of transfinite induction that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ holds for all γ.

2. (6+6+6+7=25 pts) Find the Cantor normal forms of the results of the following computations in ordinal arithmetic. (You can use **all** the identities we learned in class regarding ordinal arithmetic in Cantor normal form.)

a) $(\omega^\omega \cdot 2 + \omega^{\omega^2+\omega} \cdot 3 + \omega^{\omega^{320}} \cdot 5 + 8) + (\omega^{\omega^3} \cdot 3 + \omega^{\omega^2+\omega} \cdot 4 + 9) =$

$$= \omega^\omega \cdot 2 + \omega^{\omega^2+\omega} \cdot 3 + \omega^{\omega^{320}} \cdot 5 + \omega^{\omega^3} \cdot 3 + \omega^{\omega^2+\omega} \cdot 4 + 9$$
$$= \omega^{\omega^{320}} \cdot 5 + \omega^{\omega^3} \cdot 3 + \omega^{\omega^2+\omega} \cdot 4 + 9$$

b) $(\omega^{\omega\cdot\omega} + \omega^{\omega+1} \cdot 2 + \omega^2) + (\omega^{\omega^2} + \omega^{\omega+1}) =$

$$= \omega^{\omega\cdot\omega} + \omega^{\omega+1} \cdot 2 + \omega^2 + \omega^{\omega^2} + \omega^{\omega+1}$$
$$= \omega^{\omega\cdot\omega} + \omega^{\omega+1} \cdot 2 + \omega^{\omega^2} + \omega^{\omega+1}$$
$$= \omega^{\omega\cdot\omega} + \omega^{\omega^2} + \omega^{\omega+1}$$
$$= \omega^{\omega^2} + \omega^{\omega^2} + \omega^{\omega+1}$$
$$= \omega^{\omega^2} \cdot 2 + \omega^{\omega+1}$$

c) $(\omega^{\omega^\omega} \cdot 3 + \omega^2 \cdot 2) \cdot (\omega^{\omega^{\omega^3}} + 5) =$

$$= (\omega^{\omega^\omega} \cdot 3 + \omega^2 \cdot 2) \cdot \omega^{\omega^{\omega^3}} + (\omega^{\omega^\omega} \cdot 3 + \omega^2 \cdot 2) \cdot 5$$
$$= \omega^{\omega^\omega + \omega^{\omega^3}} + (\omega^{\omega^\omega} \cdot 15 + \omega^2 \cdot 2)$$
$$= \omega^{\omega^{\omega^3}} + \omega^{\omega^\omega} \cdot 15 + \omega^2 \cdot 2$$

d) $\epsilon_0 = \omega^{\epsilon_0}$ where $\epsilon_0 = \sup\left\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ \ldots\right\}$

Because if we exponentiate both sides in the definition of ϵ_0, we obtain

$$\omega^{\epsilon_0} = \omega^{\sup\left\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ \ldots\right\}} = \sup\left\{\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \omega^{\omega^{\omega^{\omega^\omega}}}, \ \ldots\right\} = \epsilon_0$$

3. (10 pts) Recall that ${}^3\mathbb{N}$ is the set of functions from $3 = \{0,1,2\}$ to $\mathbb{N}$. Consider the relation $\preccurlyeq$ on ${}^3\mathbb{N}$ given by

$$f \preccurlyeq g \quad \text{iff} \quad f = g \ \text{ or } \ \Big(f \neq g \text{ and } f(k) < g(k) \text{ where } k = \min\{i \in \{0,1,2\} : f(i) \neq g(i)\} \Big)$$

You are given that $({}^3\mathbb{N}, \preccurlyeq)$ is a well-ordered set. Find the order type $\theta = \operatorname{ot}({}^3\mathbb{N}, \preccurlyeq)$ of the well-ordered set $({}^3\mathbb{N}, \preccurlyeq)$ by constructing an order isomorphism $\varphi : {}^3\mathbb{N} \to \theta$.

We claim that $\theta = \omega^3$ is the order type of this well-ordered set. Consider the map $\varphi : {}^3\mathbb{N} \to \theta$ given by

$$\varphi(f) = \omega^2 \cdot f(0) + \omega \cdot f(1) + f(2)$$

for all $f \in {}^3\mathbb{N}$. Because the Cantor normal form of an ordinal is unique, φ is injective. We also have that φ is surjective because every ordinal $\alpha < \omega^3$ will have a Cantor normal form $\alpha = \omega^2 \cdot a + \omega \cdot b + c$ for some $a, b, c \in \mathbb{N}$ and consequently $\varphi(f) = \alpha$ where $f \in {}^3\mathbb{N}$ is the function with $f(0) = a,\ f(1) = b,\ f(2) = c$. Thus φ is a bijection.

To prove that φ is order preserving, it is sufficient to show $f \preccurlyeq g$ implies $\varphi(f) \leq \varphi(g)$. Let $f, g \in {}^3\mathbb{N}$. Suppose $f \preccurlyeq g$. Then, by definition, we have $f = g$, or, $f \neq g$ and $f(k) < g(k)$ where $k = \min\{i \in \{0,1,2\} : f(i) \neq g(i)\}$. We split into cases.

- **Case I:** In the case that $f = g$, we have $\varphi(f) = \varphi(g)$.
- **Case II:** In the case that $f \neq g$ and $f(k) < g(k)$ where $k = \min\{i \in \{0,1,2\} : f(i) \neq g(i)\}$, we split into three cases depending on k.
  - **Case II.a** If $k = 0$, then
  $$\varphi(f) = \omega^2 \cdot f(0) + \omega \cdot f(1) + f(2) < \omega^2 \cdot f(0) + \omega^2 = \omega^2 \cdot (f(0)+1) \leq \omega^2 \cdot g(0) \leq \omega^2 \cdot g(0) + \omega \cdot g(1) + g(2) = \varphi(g)$$
  - **Case II.b** If $k = 1$, then
  $$\varphi(f) = \omega^2 \cdot f(0) + \omega \cdot f(1) + f(2) < \omega^2 \cdot f(0) + \omega \cdot f(1) + \omega = \omega^2 \cdot f(0) + \omega \cdot (f(1)+1) \leq \omega^2 \cdot g(0) + \omega \cdot g(1) \leq \omega^2 \cdot g(0) + \omega \cdot g(1) + g(2) = \varphi(g)$$
  - **Case II.c** If $k = 2$, then
  $$\varphi(f) = \omega^2 \cdot f(0) + \omega \cdot f(1) + f(2) < \omega^2 \cdot f(0) + \omega \cdot f(1) + f(2) + 1 \leq \omega^2 \cdot g(0) + \omega \cdot g(1) + g(2) = \varphi(g)$$

  In all cases, we obtained that $\varphi(f) < \varphi(g)$

  Therefore, in both cases, we have $\varphi(f) \leq \varphi(g)$. This shows that φ is order preserving and hence is an order isomorphism.

4. (10+10+5 pts) Let ω_1 denote the first uncountable ordinal. A function $f : \omega_1 \to \omega_1$ is called *regressive* if we have $f(0) = 0$ and $f(\alpha) < \alpha$ for all $1 \leq \alpha < \omega_1$. For Part (a) and (b) of this question, fix a regressive function $f : \omega_1 \to \omega_1$. By recursion on $\mathbb{N}$, define the sets $(S_n)_{n \in \mathbb{N}}$ as follows:

$$S_0 = \{0\} \text{ and } S_{n+1} = f^{-1}[S_n] = \{\alpha \in \omega_1 : \ f(\alpha) \in S_n\}$$

For example, if it is the case that $f(\omega) = 320$ and $f(320) = 0$, then $320 \in S_1$ and $\omega \in S_2$.

a) Prove that $\omega_1 = \bigcup_{n \in \mathbb{N}} S_n$.

By definition, $\omega_1 \supseteq S_n$ for all $n \in \mathbb{N}$ and hence $\omega_1 \supseteq \bigcup_{n \in \mathbb{N}} S_n$.

To prove the other direction, let $\alpha \in \omega_1$. Observe that, since f is regressive, the sequence $(f^k(\alpha))$ is strictly decreasing, that is, $f(\alpha) > f(f(\alpha)) > f(f(f(\alpha))) > \ldots$ *unless* $f^k(\alpha) = 0$ for some $k \in \mathbb{N}$, after which point the sequence will be constantly 0. Since there is no infinite strictly decreasing sequence ordinals, there indeed must be some $k \in \mathbb{N}$ such that $f^k(\alpha) = 0$. But then $\alpha \in f^{-k}[S_0] = S_k$ and so $\alpha \in \bigcup_{n \in \mathbb{N}} S_n$. This shows that $\omega_1 \subseteq \bigcup_{n \in \mathbb{N}} S_n$.

b) Prove that there exists $\delta \in \omega_1$ such that $f^{-1}(\delta) = \{\alpha \in \omega_1 : f(\alpha) = \delta\}$ is uncountable.

Since $\omega_1 = \bigcup_{n \in \mathbb{N}} S_n$, a countable union of countable sets is countable and ω_1 is uncountable, we must have that some S_n's are uncountable. Consider the least $k \in \mathbb{N}$ for which S_k is uncountable. Clearly $k \neq 0$. Observe that

$$S_k = f^{-1}[S_{k-1}] = \bigcup_{\delta \in S_{k-1}} f^{-1}(\delta)$$

Since S_k is uncountable and the index set S_{k-1} is countable, as a countable union of countable sets is countable, we must have that $f^{-1}(\delta)$ is uncountable for some $\delta \in S_{k-1}$.

c) Construct a surjective regressive function $g : \omega_1 \to \omega_1$.

Consider the map $g : \omega_1 \to \omega_1$ given by

$$g(\alpha) = \begin{cases} \delta & \text{if } \alpha = S(\delta) \text{ for some } \delta \in \omega_1 \\ 0 & \text{if } \alpha \text{ is a limit or } \alpha = 0 \end{cases}$$

Then g is regressive because $g(0) = 0$ and for all $\alpha < \omega_1$, if α is a successor, then $S(g(\alpha)) = \alpha$ and so $g(\alpha) < \alpha$; and if α is a limit, $g(\alpha) = 0 < \alpha$. g is surjective because for any $\beta \in \omega_1$, by the definition of g, we have $g(S(\beta)) = \beta$.

5. (10 pts) Prove that for all ordinals $\alpha > 0$, if α is a limit ordinal, then $\bigcup \alpha = \alpha$.

Let $\alpha > 0$ be an ordinal. Suppose that α is a limit. We wish to show $\bigcup \alpha = \alpha$.

Let $\delta \in \bigcup \alpha$. Then, by definition, $\delta \in \beta$ for some $\beta \in \alpha$. Since α is transitive, $\beta \subseteq \alpha$ and so $\delta \in \beta \subseteq \alpha$. Hence $\bigcup \alpha \subseteq \alpha$.

Let $\delta \in \alpha$. Since α is limit ordinal, $S(\delta) \in \alpha$ because otherwise we would have $S(\delta) = \alpha$. But now, since $\delta \in S(\delta)$ and $S(\delta) \in \alpha$, by definition, $\delta \in \bigcup \alpha$. Hence $\bigcup \alpha \supseteq \alpha$.

6. (10 pts) Let ω_1 denote the first uncountable ordinal. A subset $C \subseteq \omega_1$ is called a *club* (that is, **cl**osed and **unb**ounded) subset of ω_1 if

- $\sup_{n\in\omega} \alpha_n \in C$ for every sequence $(\alpha_n)_{n\in\omega}$ over C with $\alpha_0 < \alpha_1 < \alpha_2 < \dots$ and
- For every $\alpha < \omega_1$ there exists $\beta \in C$ such that $\alpha < \beta$.

In other words, C is a club subset of ω_1 iff C contains all limit points of its strictly increasing sequences and is unbounded in ω_1. Show that the intersection of two club subsets of ω_1 is a club subset of ω_1.

Let C and D be club subsets of ω_1.

Let $(\alpha_n)_{n\in\omega}$ be a sequence over $C \cap D$ with $\alpha_0 < \alpha_1 < \alpha_2 < \dots$. In this case, trivially, α can be considered as a strictly increasing sequence over C and over D separately. But because C and D are club subsets, we have $\sup_{n\in\omega} \alpha_n \in C$ and $\sup_{n\in\omega} \alpha_n \in D$. Hence $\sup_{n\in\omega} \alpha_n \in C \cap D$.

Let $\alpha < \omega_1$. Since C is club, we can find some $\beta_0 \in C$ with $\alpha < \beta_0$. Since D is club, we can find some $\delta_0 \in D$ with $\beta_0 < \delta_0$. Using the unboundedness of these sets and continuing recursively, we can construct a sequence $(\beta_n)_{n\in\omega}$ over C and a sequence $(\delta_n)_{n\in\omega}$ over D such that

$$\alpha < \beta_0 < \delta_0 < \beta_1 < \delta_1 < \dots$$

Observe that, since the strictly increasing sequences $(\beta_n)_{n\in\omega}$ and $(\delta_n)_{n\in\omega}$ are intertwined, we must have

$$\sup_{n\in\omega} \beta_n = \theta = \sup_{n\in\omega} \delta_n$$

But since C and D are club subsets, we obtain that $\theta \in C$ and $\theta \in D$. Hence $\theta \in C \cap D$. Therefore there exists $\theta \in C \cap D$ such that $\alpha < \theta$. It follows that $C \cap D$ is a club subset of ω_1.