METU Department of Mathematics, Math 320, Final Exam, January 11, 2025, 13:30

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	7 questions on 4 pages, 120 minutes 100 points in total Justify your answer

1. (10 pts) Prove or disprove the following: For every cardinal κ, the set

$$\{\lambda : \lambda < \kappa \text{ and } \lambda \text{ is an infinite cardinal}\}$$

has cardinality less than κ.
Hint. Recall that every infinite cardinal is of the form $\aleph_\alpha$ for some ordinal α. Think whether or not it is possible to have a fixed point of the non-decreasing class function $\alpha \mapsto \aleph_\alpha$.

2. (10 pts) Let κ be an infinite cardinal. We define the generalized beth numbers $\beth_\alpha(\kappa)$ by transfinite recursion on α as follows.

$$\beth_0(\kappa) = \kappa,\ \beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)} \text{ for all ordinals } \alpha \text{ and } \beth_\gamma(\kappa) = \sup\{\beth_\theta(\kappa) : \ \theta < \gamma\} \text{ for all limit ordinals } \gamma.$$

In this notation, the standard beth number $\beth_\alpha$ is simply equal to $\beth_\alpha(\aleph_0)$.

Using transfinite induction on β, prove that $\beth_\beta\left(\beth_\alpha(\kappa)\right) = \beth_{\alpha+\beta}(\kappa)$ for all ordinals α and β.

3. (6+6+6+7=25 pts) Assuming the Generalized Continuum Hypothesis (GCH), that is, the statement $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α, find the corresponding $\aleph$ numbers of the following computations in cardinal arithmetic. (You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.)

a) $\aleph_1^{\left(\aleph_3^{\aleph_2}\right)} =$

b) $\left(\aleph_2^{\aleph_{\omega_1}} + \aleph_3 + 5\right) \cdot (\aleph_\omega^{\aleph_0} + \aleph_1) =$

c) $\left(\aleph_{\aleph_\omega}^{\aleph_0}\right)^{\aleph_\omega} =$

d) $\sum_{\alpha<\omega_1} \aleph_\alpha =$

4. (10 pts) Using König's theorem, show that $\prod_{n\in\mathbb{N}} \aleph_n > \aleph_\omega$.

5. (10+5+5 pts) Let κ be a cardinal. We define the factorial of κ by

$$\kappa! = |\mathrm{Sym}(\kappa)|$$

where $\mathrm{Sym}(\kappa) = \{\varphi : \kappa \to \kappa \mid \varphi \text{ is a bijection}\}$ is the set of bijections on κ. Under this definition, we have $0! = 1$, $3! = 6$ and the factorial behaves as traditionally expected for finite cardinals. In this question, we shall together prove that $\kappa! = 2^\kappa$ whenever κ is an infinite cardinal.

For the rest of the question, let κ be an infinite cardinal.

a) Construct an injection from ${}^\kappa 2$ to $\mathrm{Sym}(2 \times \kappa)$.

Hint. Think of the cartesian product $2 \times \kappa$ as the 2-by-κ matrix whose $n\alpha$-th entry is the ordered pair (n, α). Given a sequence $f \in {}^\kappa 2$, how could you use the data encoded in f to permute the entries of this matrix?

b) Prove that $|\mathrm{Sym}(2 \times \kappa)| \leq 2^\kappa$

c) Conclude that $\kappa! = 2^\kappa$.

6. (5+10 pts)
a) State the Continuum Hypothesis without using $\aleph$ or $\beth$ numbers.

b) Recall that the von Neumann hierarchy of sets is defined via transfinite recursion as follows.

- $\mathrm{V}_0 = \emptyset$
- $\mathrm{V}_{\alpha+1} = \mathcal{P}(\mathrm{V}_\alpha)$ for all ordinals α, and
- $\mathrm{V}_\gamma = \bigcup_{\beta<\gamma} \mathrm{V}_\beta$ for all limit ordinals γ.

Recall also that the set V_α is transitive for every α. Prove that, for any set x and for any ordinal α, $\bigcup x \in \mathrm{V}_\alpha$ implies $x \in \mathrm{V}_{\alpha+2}$.

7. (10 pts) List the axioms of ZFC. Each correctly stated axiom will earn you one point.