************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
FULL NAME	STUDENT ID	Submission deadline: 10/6/23, 10:00 4 questions on 4 pages 100 points in total

You can use all the identities and facts regarding the ordinal arithmetic listed in lecture notes and the book, unless the question itself is one of those identities, in which case you should simply prove it via the appropriate technique.

1. (10+15=25 pts) Some parts of this question are independent.

a) Let X be a non-empty set of ordinals and $\alpha > 1$ be an ordinal. Prove that

$$\sup\{\alpha^{\beta} : \beta \in X\} = \alpha^{\sup(X)}$$

b) Using transfinite induction on the appropriate variable, prove that for all ordinals α, β, γ, we have that $(\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}$

2. (10+15=25 pts) Let $\omega^{<\omega}$ denote the set of sequences over ω of finite length, that is, $\omega^{<\omega} = \{{}^n\omega : n < \omega\}$. For the purposes of this question, a subset $T \subseteq \omega^{<\omega}$ is said to be a **descriptive set-theoretic tree** if

- $T \neq \emptyset$ and
- T is closed downwards with respect to $\subseteq$, i.e. for all $s, t \in \omega^{<\omega}$, if $s \subseteq t$ and $t \in T$, then $s \in T$.

Observe that such a descriptive set-theoretic tree T can be seen as a usual graph-theoretic tree by considering the vertex set T and putting an edge between the vertices $s, t \in T$ for which we have $s \subseteq t$ and $\mathrm{dom}(t) = \mathrm{dom}(s) + 1$.

A descriptive set-theoretic tree T is said to be **well-founded** if there is no sequence $(t_n)_{n \in \mathbb{N}}$ in T such that $t_n \subseteq t_{n+1}$ and $\mathrm{dom}(t_n) = n$ for all $n \in \mathbb{N}$. In other words, T is well-founded if there is no infinite path once T is considered as a graph-theoretic tree.

Given a (descriptive set theoretic) well-founded tree T, we define the map $\rho_T : T \to \omega_1$ by recursion as follows.

$$\rho_T(t) = \sup\{\rho_T(s) + 1 : \ s \in T, \ t \subsetneq s\}$$

Such a recursion is possible due to the **well-founded recursion theorem**, since the well-foundedness of T implies that there is no sequence of vertices $t_0 \subsetneq t_1 \subsetneq t_2 \subsetneq \ldots$ in T. Using **well-founded induction**, one can show that the codomain of ρ_T is indeed ω_1. For the purposes of this question, you need not know the proofs of these facts. These are only mentioned so that you may convince yourself that ρ_T is indeed well-defined by looking these up if necessary.

The **rank** of the tree T is defined to be $\rho_T(\emptyset)$.

a) Find a well-founded tree T such that the rank of T is $\omega + 1$. You may draw a diagrammatic representation of T instead of defining T as a set.

b) Prove that for every $\alpha < \omega_1$ there exists a well-founded tree T_α such that the rank of T_α is α.

3. (15+10=25 pts) In order to understand the motivation behind this question, **please first read** the problem statements Q.1.c and Q.1.d of Take Home Exam II from Spring 2021 that you can find on my web page under "Teaching Material". For this question, you are free to use the fact that, given an ordinal $\gamma \geq 2$ and $\alpha > 0$, there exist unique ordinals $\beta_1 > \cdots > \beta_n$ and $\gamma > \delta_1, \ldots, \delta_n \geq 1$ such that

$$\alpha = \gamma^{\beta_1} \cdot \delta_1 + \cdots + \gamma^{\beta_n} \cdot \delta_n$$

In other words, using similar ideas, the existence of Cantor normal form of ordinals can be generalized to any base $\gamma \geq 2$ instead of ω.

Let α, γ be ordinals and consider the set

$$\mathrm{Fin}(\alpha, \gamma) = \{f : \alpha \to \gamma \mid f(\xi) = 0 \text{ for all but finitely many } \xi < \alpha\}$$

In other words, $\mathrm{Fin}(\alpha, \gamma)$ is the set of functions from α to γ with finite support. You are given that the relation $\prec$ on $\mathrm{Fin}(\alpha, \gamma)$ given by

$$f \prec g \text{ if and only if } f(\eta) < g(\eta) \text{ where } \eta = \max\{\xi < \alpha : \ f(\xi) \neq g(\xi)\}$$

is a strict well-order relation.

a) Prove that the order type of $(\mathrm{Fin}(\alpha, \gamma), \prec)$ is γ^α by explicitly finding an order-isomorphism from $\mathrm{Fin}(\alpha, \gamma)$ to γ^α.

b) Suppose that you are asked to prove the fact in Part (a) using transfinite induction on α. Briefly explain how the successor step would be done by providing the main idea behind the implication in this specific case.

4. (10+15=25 pts) For the first part of this question, unlike Question 2, we are using usual graph-theoretic notions. Those students who do not remember basic graph theoretic definitions should refer to their MATH112 lecture notes or any book on graph theory such as Diestel's book that was references in Take Home Exam I.

a) Prove that any connected graph has a spanning tree.

For the next part of this question, one is required to have taken MATH349. Those students who did not take this course may assume that the set X below is a compact subset of $\mathbb{R}$ in which case their topology knowledge from MATH251 would suffice.

A dynamical system is a pair (X, φ) where X is a compact metric space and $\varphi : X \to X$ is a continuous map. A subset $S \subseteq X$ is said to be **invariant** if $\varphi[S] \subseteq S$. A subset $M \subseteq X$ is said to be **minimal** if M is non-empty, closed, invariant and has no non-empty closed invariant proper subsets.

b) Let (X, φ) be a dynamical system. Show that there exists a minimal subset $M \subseteq X$.