************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: 10/5/23, 12:00 3 questions on 4 pages 100 points in total

1. (6×10=60 pts) Let X be a set and $R \subseteq X \times X$ be a relation on X. We define the *equivalence relation generated by* R to be the relation

$$\langle R \rangle = \bigcap_{\substack{R \subseteq E \\ E \text{ is an equivalence} \\ \text{relation on } X}} E$$

that is, $\langle R \rangle$ is the smallest equivalence relation on X containing R. Since an intersection of equivalence relations on X is an equivalence relation on X, the relation $\langle R \rangle$ is indeed an equivalence relation on X.

In this question, we aim to understand how $\langle R \rangle$ can be obtained using the operations of symmetric and transitive closure. The *symmetric closure* of R is defined to be the relation $R^* = R \cup R^{-1}$ and the *transitive closure* of R is defined to be the relation

$$\overline{R} = \bigcup_{n \in \mathbb{N}} R^n$$

where the sequence $(R^n)_{n \in \mathbb{N}}$ of relations on X is defined recursively as follows.

$$R^0 = \Delta_X$$
$$R^{n+1} = R \circ R^n \text{ for any } n \in \mathbb{N}$$

where $\circ$ is the composition of relations.

a) Show that R^* is a symmetric relation.

b) Show that $\overline{R}$ is a transitive relation.

c) Show that if R is symmetric, then so is $\overline{R}$.

d) Show that $\langle R\rangle = \overline{R^*} \cup \Delta_X$.

e) Suppose that $X = \mathbb{N}$ and $R = \{(n, 2n) : \; n \in \mathbb{N}\}$. Describe the equivalence class $[320]_{\langle R\rangle}$ explicitly. For this part of this question only, you need not justify your answer.

f) Suppose that $\{w \in X : \; xRw\}$ is countable for all $x \in X$. Show that $\{w \in X : \; x\overline{R}w\}$ is countable for all $x \in X$.

2. (20 pts) Consider the following proof of Kőnig's infinity lemma taken as an image from Reinhard Diestel's classic graph theory book *Graph Theory, Fifth edition. Graduate Texts in Mathematics, 173. Springer, Berlin, 2018. xviii+428 pp. ISBN: 978-3-662-57560-4; 978-3-662-53621-6.*

Lemma 8.1.2. (König's Infinity Lemma)
Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_n with $n \geqslant 1$ has a neighbour $f(v)$ in V_{n-1}. Then G contains a ray $v_0 v_1 \ldots$ with $v_n \in V_n$ for all n.

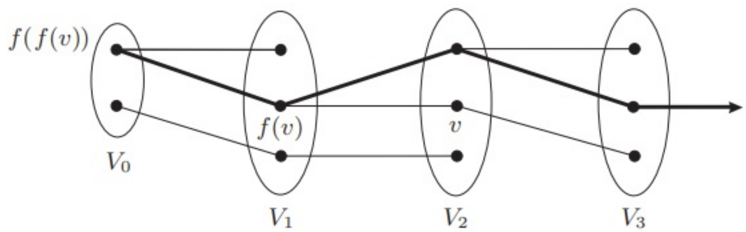

Fig. 8.1.2. König's infinity lemma

Proof. Let $\mathcal{P}$ be the set of all finite paths of the form $v\, f(v)\, f(f(v)) \ldots$ ending in V_0. Since V_0 is finite but $\mathcal{P}$ is infinite, infinitely many of the paths in $\mathcal{P}$ end at the same vertex $v_0 \in V_0$. Of these paths, infinitely many also agree on their penultimate vertex $v_1 \in V_1$, because V_1 is finite. Of those paths, infinitely many agree even on their vertex v_2 in V_2—and so on. Although the set of paths considered decreases from step to step, it is still infinite after any finite number of steps, so v_n gets defined for every $n \in \mathbb{N}$. By definition, each vertex v_n is adjacent to v_{n-1} on one of those paths, so $v_0 v_1 \ldots$ is indeed a ray. □

Your aim in this question is to justify the steps of this proof. If one were to translate this informal proof done in natural language into a formal proof in ZFC, then, together with other inference rules, axioms and theorems, one would have to specifically use *the Axiom of Separation*, *the Axiom of Choice*, *the Recursion Theorem* and *the Pigeonhole Principle* at various steps. **Briefly** explain at which steps the aforementioned axioms and theorems would be needed. You do not have to give a full list of steps at which each of these axioms and theorems are used, it is sufficient to provide one use of each of these axioms and theorems.

In the case that you need to recall the basic definitions from graph theory, you may freely preview parts of Diestel's book on its web page or officially download the book from Springer Link using the university's resources.

3. (10+10 pts) For the first part of this question, let $U \subseteq \mathbb{R} \times \mathbb{R}$ be *universal* for closed subsets of $\mathbb{R}$, that is,

$$\text{For all } C \subseteq \mathbb{R},\ C \text{ is closed if and only if } \exists x \in \mathbb{R}\ C = U_x.$$

where $U_x = \{w \in \mathbb{R} : (w, x) \in U\}$. Intuitively speaking, U is parameterizing the closed subsets of $\mathbb{R}$ since we can obtain each closed subset of $\mathbb{R}$ as a section of U by varying $x \in \mathbb{R}$.

Fun fact. Such universal sets are known to exist, not just for $\mathbb{R}$ but for arbitrary Polish spaces. Indeed, one can even take U itself to be closed, though, we shall not assume here that U is closed.

a) Show that U is not open.
Hint. What would Russell say about the set of points that are members of the sets that they are parameterizing?

For the second part of this question, let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ be an *almost disjoint* family, that is,

- A is infinite for all $A \in \mathcal{A}$ and
- $A \cap B$ is finite for all distinct $A, B \in \mathcal{A}$.

b) Show that if $\mathcal{A}$ is countably infinite, then there exists an infinite subset $S \subseteq \mathbb{N}$ such that $S \notin \mathcal{A}$ and $\mathcal{A} \cup \{S\}$ is almost disjoint.