************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	6 questions on 4 pages, 135 minutes 100 points in total Justify your answer

1. (15 pts) Recall that the addition $+$ and the multiplication $\cdot$ on ordinal numbers are recursively defined as follows.

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + S(\beta) &= S(\alpha+\beta) \\ \alpha + \gamma &= \sup\{\alpha+\theta : \theta \in \gamma\} \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot S(\beta) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \gamma &= \sup\{\alpha \cdot \theta : \theta \in \gamma\} \end{aligned}$$

for all ordinals α, β and limit ordinals γ. You are **given** that

- $+$ is associative and
- $\alpha + \sup(X) = \sup\{\alpha+\beta : \beta \in X\}$ and $\alpha \cdot \sup(X) = \sup\{\alpha \cdot \beta : \beta \in X\}$

for all α and every non-empty set of ordinals X. Using transfinite induction on the appropriate variable, prove that

$$\alpha(\beta+\delta) = \alpha \cdot \beta + \alpha \cdot \delta$$

for all ordinals α, β, δ. Show **all** your steps while referring to the relevant identity and theorem for each step.

WARNING: If you use an identity involving ordinal arithmetic other than the identities given in the question, you are supposed to prove it.

2. (5+5+5+5+5=25 pts) Assuming the Generalized Continuum Hypothesis (GCH), that is, the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for every ordinal α, find the corresponding $\aleph$ numbers of the following computations in cardinal arithmetic. You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.

a) $\aleph_{\omega+3} \cdot (2^{\aleph_{\omega+1}} + 3^{\aleph_\omega}) =$

b) $(\aleph_{\aleph_2} + \aleph_{2^{\aleph_0}}) + ((\aleph_{\aleph_0})^{320} + 1) =$

c) $(\aleph_{\omega+1})^{\aleph_0 + \aleph_\omega} =$

d) $(\aleph_{\aleph_\omega})^{\aleph_1} =$

e) $|\mathcal{P}(\mathcal{P}(\mathbb{Q})) \times \mathbb{R}|^{|{}^{\mathbb{N}}\mathbb{R}|} =$

3. (10 pts) Prove that $\mathrm{cf}(\aleph_\alpha) \leq \mathrm{cf}(\alpha)$ for all limit ordinals α.

4. (10+10 = 20 pts) Set $\mathcal{P}_{\mathrm{c}}(\omega_1) = \{S \subseteq \omega_1 : \ S \text{ is countable}\}$ and consider the poset $(\mathcal{P}_{\mathrm{c}}(\omega_1), \subseteq)$.

a) Prove that every countable chain has an upper bound in $(\mathcal{P}_{\mathrm{c}}(\omega_1), \subseteq)$ but there are uncountable chains with no upper bound in $(\mathcal{P}_{\mathrm{c}}(\omega_1), \subseteq)$.

a) A subset $\mathcal{A} \subseteq \mathcal{P}_{\mathrm{c}}(\omega_1)$ is called an **antichain** if X and Y are not comparable for any two distinct elements $X, Y \in \mathcal{A}$. Show that there are antichains of size $2^{\aleph_0}$ in $(\mathcal{P}_{\mathrm{c}}(\omega_1), \subseteq)$.

5. (5+5+10=20 pts)
a) State the Continuum Hypothesis (CH).

b) State the definition of Hartogs number. The Hartogs number of a set X is ...

c) Recall that the von Neumann hierarchy of sets is defined via transfinite recursion as follows.

- $\mathrm{V}_0 = \emptyset$
- $\mathrm{V}_{\alpha+1} = \mathcal{P}(\mathrm{V}_\alpha)$ for all ordinals α, and
- $\mathrm{V}_\gamma = \bigcup_{\beta<\gamma} \mathrm{V}_\beta$ for all limit ordinals γ.

Using transfinite induction, prove that $\alpha \in \mathrm{V}_{\alpha+1}$ for all ordinals α. (You can freely use the fact that V_α is transitive for all ordinals α.)

6. (10 pts) Recall that Cantor's theorem states that $|X| < |\mathcal{P}(X)|$ for any set X. Prove Cantor's theorem.