************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	5 questions on 5 pages 100 points in total Justify your answer

1. (8 pts) Using transfinite induction on γ, prove that for all ordinals α, β, γ if $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

2. ($4 \times 9 = 36$ pts) Let $\mathcal{I}$ denote the set of **non-decreasing** functions from ω to ω_1, that is,

$$\mathcal{I} = \{f \in {}^{\omega}\omega_1 : \ \forall i \in \omega \ f(i) \leq f(i+1)\}$$

Consider the relation E on $\mathcal{I}$ given by

$$h \ E \ k \quad \text{if and only if} \quad \sup\{h(i) : i \in \omega\} = \sup\{k(i) : i \in \omega\}$$

for all $h, k \in \mathcal{I}$.

a) Show that E is an equivalence relation.

b) Show that $\mathcal{I}/E$ is of cardinality $\aleph_1$.

c) Find $g \in \mathcal{I}$ such that $[g]_E$ is finite.

d) Find $k \in \mathcal{I}$ such that $[k]_E$ is of cardinality $2^{\aleph_0}$. **OR** d′) Find $h \in \mathcal{I}$ such that $[h]_E$ is countably infinite.

3. ($3 \times 8 = 24$ pts) Assuming the Generalized Continuum Hypothesis (GCH), that is, the statement $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α, find the corresponding $\aleph$ numbers of the following computations in cardinal arithmetic. (You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.)

a) $(\aleph_{\aleph_1})^{\aleph_{320}} =$

b) $320^{\aleph_\omega} + \aleph_\omega^{320} =$

c) $|\mathcal{P}(\mathbb{R})| \cdot |{}^{\omega_1}\omega| =$

4. (8+8 = 16 pts)

a) Prove or disprove: There exists an infinite cardinal number κ such that $\kappa = cf(\kappa)^{\kappa}$.

b) Prove or disprove: There exists an infinite cardinal number κ such that $\aleph_{\kappa} = 2^{\kappa}$.

5. (8 + 8= 16 pts) Let $f : \omega_1 \to \omega_1$ be a function such that

- $\alpha \leq f(\alpha)$ for all $\alpha \in \omega_1$, and
- $f(\sup(A)) = \sup\{f(\alpha) : \alpha \in A\}$ for all non-empty countable subsets $A \subseteq \omega_1$.

a) Prove that for all $\beta \in \omega_1$ there exists $\eta \in \omega_1$ such that $\beta \leq \eta$ and $f(\eta) = \eta$.

b) Show that the set of fixed points of f is uncountable, that is, $\{\alpha \in \omega_1 : f(\alpha) = \alpha\}$ is uncountable.