PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 80 MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(10+15+15 pts) **1.** Consider the matrix $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \in M_{3\times3}(\mathbb{R})$.

(a) Without doing any computations, explain why A is diagonalizable. Since A is a real symmetric matrix, by a theorem we learned in class, A is diagonalizable.

You are given that the eigenvalues of A are $\lambda_1 = 8$ and $\lambda_2 = -1$ and that the corresponding eigenspaces are

$$W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in M_{3\times1}(\mathbb{R}) : \ x = y = z \right\} \text{ and } W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in M_{3\times1}(\mathbb{R}) : \ x + y + z = 0 \right\}$$

(b) Find an orthogonal matrix $P \in M_{3\times3}(\mathbb{R})$ such that $P^{-1}AP$ is diagonal. Make sure that you explain why the matrix that you constructed is orthogonal. The dimensions of the eigenspaces are $\dim_{\mathbb{R}}(W_1) = 1$ and $\dim_{\mathbb{R}}(W_2) = 2$. Therefore, we wish to choose $w_1 \in W_1$ and $w_2, w_3 \in W_2$ so that $\{w_1, w_2, w_3\}$ is an orthonormal basis for $M_{3\times1}(\mathbb{R})$.

Choose, for example, $w_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ $w_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ $w_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.

Note. These vectors are not unique and can be chosen in the following way: Pick an arbitrary vector in W_1 and normalize it to choose w_1. Pick two linearly independent vectors in W_2, apply Gram-Schmidt to orthogonalize them and then, normalize them to choose w_2, w_3. Now, set $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$. Since the columns of P form an orthonormal set of vectors, P is orthogonal. Moreover, $P^{-1}AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) Find a cube root of A, that is, find a matrix $B \in M_{3\times3}(\mathbb{R})$ such that $B^3 = A$.

Set $B = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1}$. Then, by Part (b), $B^3 = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^3 P^{-1} = A$.

(15 pts) 2. Recall that every matrix $A \in M_{3\times 3}(\mathbb{C})$ defines a (sesqui-linear) form f_A on $M_{3\times 1}(\mathbb{C})$ via the rule $f(X,Y) = Y^*AX$. Complete the blanks appropriately so that the form f_B defined by the matrix $B \in M_{3\times 3}(\mathbb{C})$ is a positive form, that is, an inner product.

$$M = \begin{bmatrix} 1/2 & 2i & 0 \\ -2i & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remark. For this question only, there shall be no partial credits and you shall receive full or no points for each blank. Hence, you need *not* show your computation but only need to fill in the blanks with appropriate scalars.

(15+15+15 pts) 3. Consider the set $V = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists m \in \mathbb{N}\ \forall n \geq m\ a_n = 0\}$ of real-valued sequences that are eventually zero. Then V is an inner product space over $\mathbb{R}$ where the vector addition and the scalar multiplication are usual component-wise addition and multiplication, and the inner product is given by

$$\langle (a_n)_{n\in\mathbb{N}} \mid (b_n)_{n\in\mathbb{N}} \rangle = \sum_{n=0}^{\infty} a_n b_n$$

Consider the left-shift operator $L : V \to V$ given by $L\big((a_n)_{n\in\mathbb{N}}\big) = (a_{n+1})_{n\in\mathbb{N}}$. In other words, we have $L(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots)$.

(a) You are given that L^* exists. Find L^* and verify that the linear operator you propose is indeed L^* using the definition of adjoint. By definition, L^* is the unique linear operator on V such that $\langle L((a_n)_{n\in\mathbb{N}}) \mid (b_n)_{n\in\mathbb{N}} \rangle = \langle (a_n)_{n\in\mathbb{N}} \mid L^*((b_n)_{n\in\mathbb{N}}) \rangle$ for all $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V$. Consider the right-shift operator $R : V \to V$ given by $R(a_0, a_1, a_2, a_3, \dots) = (0, a_0, a_1, a_2, \dots)$. Then R is a linear operator on V and moreover, $\langle L((a_n)_{n\in\mathbb{N}}) \mid (b_n)_{n\in\mathbb{N}} \rangle = \sum_{n=0}^{\infty} a_{n+1}b_n = a_1b_0 + a_2b_1 + \cdots = a_0 0 + a_1 b_0 + a_2 b_1 + \cdots = \langle (a_n)_{n\in\mathbb{N}} \mid (0, b_0, b_1, b_2, \dots) \rangle = \langle (a_n)_{n\in\mathbb{N}} \mid R((b_n)_{n\in\mathbb{N}}) \rangle$ for all $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V$. Thus $L^* = R$.

Remark. Although we defined normality of a linear operator for finite dimensional inner product spaces, that definition indeed extends to arbitrary inner product spaces as long as the adjoint of the operator exists.

(b) Determine whether or not L is normal. Observe that

$$LL^*(1,0,0,0,\dots) = L(L^*(1,0,0,0,\dots)) = L(0,1,0,0,\dots) = (1,0,0,0,\dots)$$

$$L^*L(1,0,0,0,\dots) = L^*(L(1,0,0,0,\dots)) = L^*(0,0,0,0,\dots,) = (0,0,0,0,\dots)$$

Thus $LL^* \neq L^*L$ on V. By definition, this means that L is not normal.

Consider $U : V \to V$ given by $U(a_0, a_1, a_2, a_3, a_4, a_5, \dots) = (a_1, a_0, a_3, a_2, a_5, a_4 \dots)$, that is, given a sequence $\mathbf{a} \in V$, the map U permutes the entries a_{2k} and a_{2k+1} for each $k \in \mathbb{N}$. You are given that U is a vector space isomorphism.

(c) Show that U is a self-adjoint unitary operator. We have $\langle U((a_n)_{n\in\mathbb{N}}) \mid (b_n)_{n\in\mathbb{N}} \rangle = \langle (a_1, a_0, a_3, a_2, a_5, a_4 \dots) \mid (b_n)_{n\in\mathbb{N}} \rangle = a_1b_0 + a_0b_1 + a_3b_2 + b_2a_3 + \cdots = a_0b_1 + a_1b_0 + b_2a_3 + a_3b_2 + \cdots = \langle (a_n)_{n\in\mathbb{N}} \mid (b_1, b_0, b_3, b_2, b_5, b_4 \dots) \rangle = \langle (a_n)_{n\in\mathbb{N}} \mid U((b_n)_{n\in\mathbb{N}}) \rangle$ for all $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V$. Observe that, the terms in the sum can be rearranged in the third equality because this infinite sum is indeed a finite sum due to terms being eventually zero. Now, by definition of adjoint, we have $U = U^*$, that is, U is self-adjoint. Observe that $U^2 = I$. But then, $UU^* = U^*U = U^2 = I$ which implies that U is a unitary operator. (Alternatively, since U is given to be a vector space isomorphism, one can show that U is unitary by showing that it preserves inner product.)