PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION
Jørgen Pedersen Gram	**1850-1916**	80 MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

(13+15+15+15+15 pts) **1.** Consider the set $\mathbb{R}[x]$ of polynomials with coefficients in $\mathbb{R}$ as a vector space over $\mathbb{R}$ together with the standard operations.
For each $n \in \mathbb{N}^+$, consider the subspace $P_n = \{f \in \mathbb{R}[x] : \deg(f) < n\} \cup \{\mathbf{0}\}$ with $\dim_{\mathbb{R}}(P_n) = n$ and let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ be the differentiation operator.

(a) Let $f \in \mathbb{R}[x]$ be non-zero. Show that the set $S_f = \{D^k(f) : 0 \le k \le \deg(f)\}$ is linearly independent.

Set $m = \deg(f)$, say, $f(x) = \sum_{i=0}^{m} \beta_i x^i$. Observe that for each $0 \le k \le m$, we have $\deg(D^k(f)) = m - k$ and the coefficient of x^{m-k} in $D^k(f)$ is equal to $\binom{m}{k} k! \beta_m$. Now let $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ and suppose that $\alpha_0 D^0(f) + \cdots + \alpha_m D^m(f) = 0$. By our observation on degrees, the coefficient of x^m on the left hand side equals $\alpha_0 \beta_m$. Since $\beta_m \neq 0$, we have that $\alpha_0 = 0$. It follows that $\alpha_1 D^1(f) + \cdots + \alpha_m D^m(f) = 0$. Similarly, the coefficient of x^{m-1} on the left hand side equals $\alpha_1 m \beta_m$. Since $m\beta_m \neq 0$, we have that $\alpha_1 = 0$. It follows that $\alpha_2 D^2(f) + \cdots + \alpha_m D^m(f) = 0$. Continuing in this fashion inductively, we obtain that $\alpha_0 = \cdots = \alpha_m = 0$. Therefore $D^0(f), \dots, D^m(f)$ are linearly independent.

(b) Prove that any D-invariant subspace of $\mathbb{R}[x]$ with dimension $n \in \mathbb{N}^+$ is equal to P_n.

Let $n \in \mathbb{N}^+$ and let $V \subseteq \mathbb{R}[x]$ be a D-invariant subspace of dimension n. Let $0 \neq f \in V$. We shall show that $\deg(f) < n$. Since V is D-invariant, $S_f = \{D^0(f), \dots, D^m(f)\} \subseteq V$ where $m = \deg(f)$. By Part (a), S_f is linearly independent. Since V is of dimension n, we must have $m + 1 = |S_f| \le n$ and hence $m < n$. It follows that $V \subseteq P_n$. However, both subspaces V and P_n have dimension n and hence, $V = P_n$.

In the rest of this question, consider $\mathbb{R}[x]$ as an inner product space over $\mathbb{R}$ with the inner product $\langle f \mid g \rangle = \displaystyle\int_{-1}^{1} f(x)g(x)dx$. Let $W = \operatorname{span}(\{x^{2n+1} : n \in \mathbb{N}\}) = \langle x, x^3, x^5, \dots \rangle$.

(c) Let $k, \ell \in \mathbb{N}$. Show that x^k and x^ℓ are orthogonal if and only if k and ℓ have different parity. Using this fact, conclude that $D(W) \subseteq W^\perp$.

Suppose that k and ℓ have different parity. Then $k + \ell$ is odd and $x^{k+\ell}$ is an odd function. It follows that $0 = \int_{-1}^{1} x^{k+\ell} dx = \int_{-1}^{1} x^k x^\ell dx = \langle x^k \mid x^\ell \rangle$ and hence, x^k and x^ℓ are orthogonal. Now suppose that k and ℓ have the same parity. Then $k + \ell$ is even. It follows that $\frac{2}{k+\ell+1} = \int_{-1}^{1} x^{k+\ell} dx = \int_{-1}^{1} x^k x^\ell dx = \langle x^k \mid x^\ell \rangle$ and hence, x^k and x^ℓ are not orthogonal. Since $W = \langle x, x^3, x^5, \dots \rangle$, we have that $D(W) = \langle D(x), D(x^3), D(x^5), \dots \rangle = \langle 1, 3x^2, 5x^4, \dots \rangle = \langle 1, x^2, x^4, \dots \rangle$. Let $g \in D(W)$, say, $g(x) = \sum_{i=0}^{m} \beta_i x^{2i}$. Then, for any $f \in W$, say, $f(x) = \sum_{j=0}^{n} \alpha_j x^{2j+1}$, we have $\langle g \mid f \rangle = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_j \beta_i \langle x^{2i} \mid x^{2j+1} \rangle = 0$. Hence g is orthogonal to every vector in W. It follows that $D(W) \subseteq W^\perp$

(d) Show that $\mathbb{R}[x] = W \oplus D(W)$. Using this fact, conclude that $D(W) = W^{\perp}$.
Let $f \in \mathbb{R}[x]$, say, $f(x) = \sum_{i=0}^{n} \alpha_i x^i$. Consider $f_{\text{odd}}(x) = \sum_{0\le 2i+1\le n} \alpha_{2i+1}x^{2i+1}$ and $f_{\text{even}}(x) = \sum_{0\le 2i\le n} \alpha_{2i}x^{2i}$. Observe that $f_{\text{odd}} \in W$ and $f_{\text{even}} \in D(W)$. Moreover, we have $f(x) = f_{\text{odd}}(x) + f_{\text{even}}(x)$. It follows that $\mathbb{R}[x] = W + D(W)$. On the other hand, $W \cap D(W) = \{0\}$ since there is no non-zero polynomial whose monomials are both even and odd degree. Thus $\mathbb{R}[x] = W \oplus D(W)$.
For the conclusion, let $g \in W^{\perp}$. As above, we can write g as $g(x) = g_{\text{odd}}(x) + g_{\text{even}}(x)$ where $g_{\text{odd}} \in W$ and $g_{\text{even}} \in D(W)$. Since $g_{\text{odd}} \in W$ and $g \in W^{\perp}$, we have

$$0 = \langle g_{odd} \mid g\rangle = \langle g_{odd} \mid g_{odd} + g_{even}\rangle = \langle g_{odd} \mid g_{odd}\rangle + \langle g_{odd} \mid g_{even}\rangle = \langle g_{odd} \mid g_{odd}\rangle = \|g_{odd}\|^2$$

and hence $g_{\text{odd}} = 0$. It follows that $g = g_{\text{even}} \in D(W)$. This shows that $W^{\perp} \subseteq D(W)$. We already proved $D(W) \subseteq W^{\perp}$ in Part (b). Consequently, $D(W) = W^{\perp}$.

(e) Show that we have $\left(\int_{-1}^{1} x^{260} f(x)dx\right)^2 + \left(\int_{-1}^{1} x^{261} f(x)dx\right)^2 \le 262 \int_{-1}^{1} f(x)^2 dx$ for every $f \in \mathbb{R}[x]$. By Part (b), the set $\{x^{260}, x^{261}\}$ is orthogonal and hence, by Bessel's inequality, for every $f \in \mathbb{R}[x]$, we have $\frac{|\langle f|x^{260}\rangle|^2}{\|x^{260}\|^2} + \frac{|\langle f|x^{261}\rangle|^2}{\|x^{261}\|^2} \le \|f\|^2$. By an easy calculation, we have that $\|x^{260}\|^2 = \frac{2}{521}$ and $\|x^{261}\|^2 = \frac{2}{523}$. Therefore, the inequality above implies that

$$\frac{\left(\int_{-1}^{1} x^{260} f(x)dx\right)^2}{2/521} + \frac{\left(\int_{-1}^{1} x^{261} f(x)dx\right)^2}{2/523} \le \|f\|^2$$

Thus $\left(\int_{-1}^{1} x^{260} f(x)dx\right)^2 + \left(\int_{-1}^{1} x^{261} f(x)dx\right)^2 \le \|f\|^2 = \int_{-1}^{1} f(x)^2dx \le 262 \int_{-1}^{1} f(x)^2 dx.$

***(12 pts)* 2.** Consider $\mathbb{R}^3$ with the standard inner product. Fill in the blanks below to determine vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ so that an application of the Gram-Schmidt orthogonalization process **to the vectors** $v_1, v_2, v_3 \in \mathbb{R}^3$ in this order results in the vectors $(1,1,1), (-1,0,1), (1,-2,1) \in \mathbb{R}^3$.

Remark. For this question only, there shall be no partial credits and hence, you need *not* show your computation but only need to fill in the blanks with appropriate scalars.

$$v_1 = (1,1,1)$$
$$v_2 = (-1,0,1) + (5,5,5)$$
$$v_3 = (1,-2,1) + (2,6,10)$$

Recall the following basic facts: Any odd degree polynomial in $\mathbb{R}[x]$ has a root in $\mathbb{R}$ and any quadratic polynomial polynomial in $\mathbb{R}[x]$ with no roots in $\mathbb{R}$ has distinct roots in $\mathbb{C}$.

***(15 pts)* 3.** Let $A \in M_{3\times 3}(\mathbb{R})$ be **not** trianguable over $\mathbb{R}$. Show that A is diagonalizable over $\mathbb{C}$. Consider the characteristic polynomial $p(x)$ and the minimal polynomial $m(x)$ of the matrix A. Since $\deg(p(x)) = 3$, the polynomial $p(x)$ has at least one root in $\mathbb{R}$. If $p(x)$ had more than one root in $\mathbb{R}$, then $p(x)$ would be a product of linear factors in $\mathbb{R}[x]$ and, since $m(x) \mid p(x)$ by the Cayley-Hamilton theorem, $m(x)$ would a product of linear factors in $\mathbb{R}[x]$, implying that A is trianguable over $\mathbb{R}$, which we know is not the case. Thus $p(x)$ has exactly one root in $\mathbb{R}$. But then, by the given fact, $p(x)$ must have three distinct roots in $\mathbb{C}$. It now follows from $m(x) \mid p(x)$ that $m(x)$ is a product of **distinct** linear factors in $\mathbb{C}[x]$ and hence A is diagonalizable over $\mathbb{C}$.