PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION
ARTHUR CAYLEY	1821-1895	80 MINUTES
2 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

(15+13 pts) **1.** Consider the set $V = \{(a_n)_{n\in\mathbb{N}} : \ \forall n \in \mathbb{N}\ a_n \in \mathbb{R} \text{ and } \lim_{n\to\infty} a_n = 0\}$ of real-valued sequences converging to 0. Then V is a vector space over $\mathbb{R}$ where the vector addition and the scalar multiplication are given by

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}} \quad \text{and} \quad \alpha \cdot (a_n)_{n\in\mathbb{N}} = (\alpha a_n)_{n\in\mathbb{N}}$$

Consider the linear operator $T : V \to V$ given by $T\big((a_n)_{n\in\mathbb{N}}\big) = (a_{n+1})_{n\in\mathbb{N}}$. In other words, we have $T(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots)$.

(a) Prove that if $\lambda \in \mathbb{R}$ is an eigenvalue of T, then $|\lambda| < 1$.

Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of T, say, $T(a_0, a_1, a_2 \dots) = \lambda \cdot (a_0, a_1, a_2 \dots)$ for some eigenvector $\mathbf{a} = (a_0, a_1, \dots) \in V$. Then we have $(a_1, a_2, a_3, \dots) = (\lambda a_0, \lambda a_1, \lambda a_2, \dots)$ and hence $a_{n+1} = \lambda a_n$ for all $n \in \mathbb{N}$. It now follows from induction that $a_n = \lambda^n a_0$ for all $n \in \mathbb{N}$. Since $\mathbf{a} \in V$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \lambda^n a_0 = a_0 \lim_{n\to\infty} \lambda^n = 0$$

As $\mathbf{a}$ is an eigenvector, we have $\mathbf{a} \neq \mathbf{0}$ and this implies that $a_0 \neq 0$. Consequently, $\lim_{n\to\infty} \lambda^n = 0$. If it were that $|\lambda| \geq 1$, we would have that $\lim_{n\to\infty} \lambda^n \neq 0$. Therefore $|\lambda| < 1$.

(b) Prove that $f(T) \neq \mathbf{0}$ for any non-zero polynomial $f \in \mathbb{R}[x]$, where $\mathbf{0}$ is the zero operator on V. Explain why this does not contradict the existence of minimal polynomial of a linear operator.

Observe that $T^k(a_0, a_1, a_2, \dots) = (a_k, a_{k+1}, a_{k+2}, \dots)$ for all $k \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^n \alpha_k x^k \in \mathbb{R}[x]$ be a non-zero polynomial. Then, for all $(a_0, a_1, a_2, \dots) \in V$, we have that

$$f(T)(a_0, a_1, a_2, \dots) = \left(\sum_{k=0}^n \alpha_k T^k\right)(a_0, a_1, a_2, \dots) = \sum_{k=0}^n \alpha_k (a_k, a_{k+1}, a_{k+2}, \dots)$$

and consequently, the first entry of $f(T)(a_0, a_1, a_2, \dots)$ equals $\sum_{k=0}^n \alpha_k a_k$. Now consider the sequence $(\alpha_0, \alpha_1, \dots, \alpha_n, 0, 0, \dots) \in V$. Then $f(T)(\alpha_0, \alpha_1, \dots, \alpha_n, 0, 0, \dots) \neq \mathbf{0}$ since its first entry equals $\sum_{k=0}^n \alpha_k^2 > 0$. Thus $f(T) \neq \mathbf{0}$. This does not contradict the existence of minimal polynomial because we have proven that such polynomials exist in the finite dimensional case. However, V is infinite dimensional (and it is not necessary that the ideal of polynomials annihilating T is non-zero in this case.)

(12 pts) **2.** Let V be a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$ with two elements and let $T : V \to V$ be a linear operator on V. Determine whether or not the set $\{f \in \mathbb{Z}_2[x] : \ f^2(T) = \mathbf{0}\}$ is an ideal of $\mathbb{Z}_2[x]$, where $f^2(T) = (f \cdot f)(T) = f(T) \circ f(T)$.

We shall prove that the set $S = \{f \in \mathbb{Z}_2[x] : \ f^2(T) = \mathbf{0}\}$ is an ideal. First, observe that $S \neq \emptyset$ since S contains the zero polynomial. Let $f, g \in S$. Since $(-f)^2 = f^2$, we have that $(-f)^2(T) = f^2(T) = \mathbf{0}$ and so $-f \in S$. Moreover, since $2 = 0$ in $\mathbb{Z}_2$ and $f^2(T) = g^2(T) = \mathbf{0}$, we have that $(f+g)^2 = f^2 + 2fg + g^2 = f^2 + g^2$ and that $(f+g)^2(T) = (f^2 + g^2)(T) = f^2(T) + g^2(T) = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus $f + g \in S$. This shows that S is a subgroup of $(\mathbb{Z}_2[x], +)$. Now let $f \in S$ and $g \in \mathbb{Z}_2[x]$. Then, since $f^2(T) = \mathbf{0}$, we have that

$$(g \cdot f)^2(T) = (f \cdot g)^2(T) = (f^2 \cdot g^2)(T) = f^2(T) \circ g^2(T) = \mathbf{0} \circ g^2(T) = \mathbf{0}$$

Hence $f \cdot g = g \cdot f \in S$. It follows that S is an ideal.

***(15+15+15+15 pts)* 3.** Consider the matrix $A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ -2 & -2 & 2 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$.

(a) Show that the eigenvalues of the matrix A are $\lambda_1 = 2$ and $\lambda_2 = 6$ by finding its characteristic polynomial.

The characteristic polynomial is

$$\det(xI - A) = \det \begin{bmatrix} x-4 & -2 & 0 \\ -2 & x-4 & 0 \\ 2 & 2 & x-2 \end{bmatrix} = (x-4)\det \begin{bmatrix} x-4 & 0 \\ 2 & x-2 \end{bmatrix} - (-2)\det \begin{bmatrix} -2 & 0 \\ 2 & x-2 \end{bmatrix}$$

$$= (x-4)^2(x-2) + 2(-2)(x-2) = (x-2)(x^2 - 8x + 16 - 4)$$

$$= (x-2)(x^2 - 8x + 12) = (x-2)^2(x-6)$$

So the eigenvalues, being the roots of the characteristic polynomial, are $\lambda_1 = 2$ and $\lambda_2 = 6$.

(b) Find the dimensions of the eigenspaces W_1 and W_2 corresponding to the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 6$ respectively.

We need to find the row reduced echelon forms of the matrices $2I - A$ and $6I - A$. Applying Gaussian elimination for $2I - A$, we obtain

$$2I - A = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow[-R_1+R_2\to R_2,\ R_1+R_3\to R_3,\ -1/2R_1\to R_1]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the rank of this matrix is 1, the corresponding eigenspace $W_1 = \{(x, y, z) \in \mathbb{R}^3 : \ x + y = 0\}$ has dimension 2. Choosing two linearly independent elements of W_1, we can pick a basis for W_1 as $\mathcal{B}_1 = \{(1, -1, 0), (0, 0, 1)\}$. Applying the same procedure for $6I - A$, we obtain

$$6I - A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 2 & 2 & 4 \end{bmatrix} \xrightarrow[R_1+R_2\to R_2,\ -R_1+R_3\to R_3,\ 1/2R_1\to R_1,\ 1/4R_3\to R_3,\ R_3+R_1\to R_1]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Since the rank of this matrix is 2, the corresponding eigenspace $W_2 = \{(x, y, z) \in \mathbb{R}^3 : \ x+z = 0,\ y+z = 0\}$ has dimension 1. Choosing a vector in W_2, we obtain that $\mathcal{B}_2 = \{(1, 1, -1)\}$ is a basis for W_2.

Note. It is not necessary to find explicit bases for these eigenspaces in this part to find the dimensions of these eigenspaces. However, since we shall need these in Part (c), we found them here.

(c) If exists, find a matrix $P \in M_{3\times 3}(\mathbb{R})$ such that $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. If such a matrix does not exist, explain why this is the case.

Since the dimensions of the eigenspaces $2 + 1 = 3$ add up to the dimension of the underlying vector space $\mathbb{R}^3$, we know that A is diagonalizable. Indeed, choosing the rows of P as linearly independent eigenvectors corresponding to 2, 6 and 2, for example, $P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, we will have $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(d) Find the minimal polynomial of A. First, we know that the minimal polynomial must have all eigenvalues as its roots. Moreover, by the Cayley-Hamilton theorem, we know that the minimal polynomial must divide the characterisctic polynomial $(x-2)^2(x-6)$. Therefore the minimal polynomial equals either $f(x) = (x-2)(x-6)$ or $g(x) = (x-2)^2(x-6)$. Since

$$P^{-1}f(A)P = f\left(P^{-1}AP\right) = f\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we must have that $f(A)$ is the zero matrix, and hence the minimal polynomial is $f(x) = (x-2)(x-6)$.