PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 140 MINUTES
4 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

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(8+6+6+10+10+10=50 pts) **1.** In some parts of this question, you will be given some information. While answering each part, you are supposed to only use the information given *including and up to* that part. Let $A \in M_{4\times 4}(\mathbb{R})$.

(a) Suppose that the characteristic polynomial of A is $p(x) = (x-2)^2(x-6)^2$. List all four different possibilities for the Jordan normal form J of A that are different up to changing the places of the relevant Jordan blocks.

$$J_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad J_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad J_3 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad J_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

(b) Suppose furthermore that $\dim_{\mathbb{R}}(\ker(A-2I)) = 1$. Under this additional assumption, which of the four possibilities in Part (a) could be the Jordan normal form J of A?

Since $\dim_{\mathbb{R}}(\ker(A-2I)) = \dim_{\mathbb{R}}(\ker(J-2I))$, we check the relevant dimensions. After computing the row ranks of relevant matrices, we see that $\dim_{\mathbb{R}}(\ker(J_1-2I)) = \dim_{\mathbb{R}}(\ker(J_2-2I)) = 2$ and $\dim_{\mathbb{R}}(\ker(J_3-2I)) = \dim_{\mathbb{R}}(\ker(J_4-2I)) = 1$. Hence, among the four possibilities, only J_3 and J_4 could be the Jordan form.

(c) Suppose furthermore that $\dim_{\mathbb{R}}(\ker(A-6I)) = 2$. If possible, determine the Jordan normal form J of A. If it is not possible to determine J uniquely with the given information, explain why.

As in Part (b), we observe that $\dim_{\mathbb{R}}(\ker(A-6I)) = \dim_{\mathbb{R}}(\ker(J-6I))$ and compare the relevant dimensions. Since $\dim_{\mathbb{R}}(\ker(J_3-6I)) = 2$ and $\dim_{\mathbb{R}}(\ker(J_4-6I)) = 1$, we obtain that the Jordan form J is uniquely determined as $J = J_3$.

For the rest of this question, suppose that $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 4 & -1 & 6 & 0 \\ 0 & 4 & 0 & 6 \end{bmatrix}$. Then A satisfies all assumptions in Part (a)-(c) and has the Jordan normal form J.

Let W_1 and W_2 be the eigenspaces of A corresponding its eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 6$ respectively.

(d) Find a basis for the eigenspace W_1.

Applying Gaussian elimination, we obtain $A - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -1 & 4 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which corresponds to the system $x + z = 0,\ y = 0,\ t = 0$ and hence we have $W_1 = \left\{ \begin{bmatrix} x \\ 0 \\ -x \\ 0 \end{bmatrix} \in M_{4\times 1}(\mathbb{R}) :\ x \in \mathbb{R} \right\}$. A basis for this subspace is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

You are given that $W_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \\ t \end{bmatrix} \in M_{4\times 1}(\mathbb{R}) :\ z, t \in \mathbb{R} \right\}$.

(e) Find $P \in M_{4\times 4}(\mathbb{R})$ such that $J = P^{-1}AP$. By a theorem, we know that such an invertible P must exist. Let $v_1, v_2, v_3, v_4 \in M_{4\times 1}(\mathbb{R})$ be the columns of P. Then the equality $PJ = AP$ gives $Av_1 = 2v_1,\ Av_2 = v_1 + 2v_2,\ Av_3 = 6v_3,\ Av_4 = 6v_4$. Therefore v_3, v_4 must be linearly independent eigenvectors for $\lambda_2 = 6$ and v_1 must be an eigenvector for $\lambda_1 = 2$ so that the equation $Av_2 = v_1 + 2v_2$ has a solution for v_2. Having computed the eigenspaces, choose $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. We wish to be able to solve for $(A - 2I)v_2 = v_1$. Applying Gaussian elimination to the augmented system, we obtain that $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is a solution. Therefore $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ is as desired.

It follows from the Jordan-Chevalley decomposition theorem that A can be written as $A = D + N$ where D is a diagonalizable matrix and N is a nilpotent matrix.

(f) Find D and N, possibly by expressing them in terms of P and other matrices. Make sure that you explain why the matrices D and N you propose are diagonalizable and nilpotent, respectively.

Set $D = P\widehat{D}P^{-1}$ and $N = P\widehat{N}P^{-1}$ where $\widehat{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ and $\widehat{N} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Then D is diagonalizable since P is invertible and N is nilpotent since $N^2 = P\widehat{N}^2P^{-1} = P\mathbf{0}_{4\times 4}P^{-1} = \mathbf{0}_{4\times 4}$. Moreover, since $J = \widehat{D} + \widehat{N}$ and $J = P^{-1}AP$, we have that

$$A = PJP^{-1} = P(\widehat{D} + \widehat{N})P^{-1} = P\widehat{D}P^{-1} + P\widehat{N}P^{-1} = D + N$$

***(10+5+5=20 pts)* 2.** Consider the linear operator $T : M_{2\times 1}(\mathbb{R}) \to M_{2\times 1}(\mathbb{R})$ given by $T(X) = MX$ where $M = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ and $M_{2\times 1}(\mathbb{R})$ is endowed with its standard inner product $\langle X \mid Y \rangle = Y^*X$. Then T is a self-adjoint nonnegative operator on $M_{2\times 1}(\mathbb{R})$ and has a spectral resolution. You are given that the eigenvalues of T are $\lambda_1 = 3$ and $\lambda_2 = 5$ and the corresponding eigenspaces are

$$W_1 = \left\{ \begin{bmatrix} x \\ -x \end{bmatrix} \in M_{2\times 1}(\mathbb{R}) : \ x \in \mathbb{R} \right\} \text{ and } W_2 = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \in M_{2\times 1}(\mathbb{R}) : \ x \in \mathbb{R} \right\}.$$

(a) Find the linear operators $E_1 : M_{2\times 1}(\mathbb{R}) \to M_{2\times 1}(\mathbb{R})$ and $E_2 : M_{2\times 1}(\mathbb{R}) \to M_{2\times 1}(\mathbb{R})$ such that $T = 3E_1 + 5E_2$ is the spectral resolution of T.

We know from the Spectral Theorem that E_1 and E_2 are the orthogonal projection maps onto W_1 and W_2 respectively. Since W_1 is 1-dimensional, the orthogonal projection E_1 can be computed by projecting a vector along the unit vector $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \in W_1$. It follows that

$$E_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \left\langle \begin{bmatrix} a \\ b \end{bmatrix} \mid \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} \\ -\frac{a-b}{2} \end{bmatrix}$$

Similarly, since W_2 is 1-dimensional, the orthogonal projection E_2 can be computed by projecting a vector along the unit vector $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \in W_2$ and hence

$$E_2\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \left\langle \begin{bmatrix} a \\ b \end{bmatrix} \mid \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{bmatrix}$$

(b) Find the linear operator $S : M_{2\times 1}(\mathbb{R}) \to M_{2\times 1}(\mathbb{R})$ that is the square-root of the nonnegative operator T, that is, $S^2 = T$.

Set $S = \sqrt{3}E_1 + \sqrt{5}E_2$. Then, since $E_1^2 = E_1$, $E_2^2 = E_2$ and $E_1E_2 = E_2E_1 = \mathbf{0}$, we have

$$S^2 = (\sqrt{3}E_1 + \sqrt{5}E_2)^2 = 3E_1^2 + \sqrt{3}\sqrt{5}E_1E_2 + \sqrt{3}\sqrt{5}E_2E_1 + 5E_2^2 = 3E_1 + 5E_2 = T$$

(c) Explicitly find a matrix $N \in M_{2\times 2}(\mathbb{R})$ such that $N^2 = M$.

Consider $N = [S]_{\mathcal{B}}$ where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the standard ordered basis. Then we know

$$N^2 = [S]_{\mathcal{B}}^2 = \left[S^2\right]_{\mathcal{B}} = [T]_{\mathcal{B}} = M$$

Computing N explicitly, we obtain

$$N = \begin{bmatrix} \dfrac{\sqrt{3}+\sqrt{5}}{2} & \dfrac{-\sqrt{3}+\sqrt{5}}{2} \\ \dfrac{-\sqrt{3}+\sqrt{5}}{2} & \dfrac{\sqrt{3}+\sqrt{5}}{2} \end{bmatrix}$$

(7+8 pts) **3.** **(a)** Disprove the following: For any inner product space V and for any subspaces $W, Z \subseteq V$, if $V = W \oplus Z$, then $Z = W^\perp$.

We provide a counterexample. Consider $V = \mathbb{R}^2$ with its standard inner product and choose $W = \{(x, 0) : x \in \mathbb{R}\}$ and $Z = \{(x, x) : x \in \mathbb{R}\}$. Then we have $V = W \oplus Z$ however $Z \neq W^\perp = \{(0, x) : x \in \mathbb{R}\}$.

(b) Prove the following: For any inner product space V and for any subspaces $W, Z \subseteq V$, if $V = W \oplus Z$ and $Z \subseteq W^\perp$, then $Z = W^\perp$.

Let V be an inner product space and $W, Z \subseteq V$ be subspaces. Assume that $V = W \oplus Z$ and $Z \subseteq W^\perp$. Since we already have $Z \subseteq W^\perp$, we wish to prove $W^\perp \subseteq Z$. Let $\widehat{w} \in W^\perp$. Since $V = W \oplus Z$, $\widehat{w} = w + z$ for some $w \in W$ and $z \in Z \subseteq W^\perp$. Since $\widehat{w}, z \in W^\perp$, we have

$$0 = \langle w \mid \widehat{w} \rangle = \langle w \mid w + z \rangle = \langle w \mid w \rangle + \langle w \mid z \rangle = \langle w \mid w \rangle$$

It follows that $w = 0$ and hence $\widehat{w} = w + z = z \in Z$.

(7+8 pts) **4.** For the parts of this question, similar versions of which were asked as midterm questions during the semester, *no partial credit will be given unless the answer is detailed and completely correct.*

(a) Let $A \in M_{2\times 2}(\mathbb{R})$ be **not** triangulable over $\mathbb{R}$. Show that A is diagonalizable over $\mathbb{C}$.

Let $m(x)$ be the minimal polynomial of A. Since A is not triangulable over $\mathbb{R}$, $m(x)$ is not a product of linear factors in $\mathbb{R}[x]$. But then, since $\deg(m(x)) \leq 2$, it must be a quadratic irreducible polynomial over $\mathbb{R}$. It follows that $m(x) = (x - \alpha)(x - \beta)$ for some **distinct** $\alpha, \beta \in \mathbb{C}$. (Indeed, we must have $\overline{\alpha} = \beta$.) It follows that A is diagonalizable over $\mathbb{C}$.

(b) Consider the vector space $V = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists m \in \mathbb{N}\ \forall n \geq m\ a_n = 0\}$ together with the inner product given by

$$\langle (a_n)_{n\in\mathbb{N}} \mid (b_n)_{n\in\mathbb{N}} \rangle = \sum_{n=0}^{\infty} a_n b_n$$

Consider $U : V \to V$ given by $U(a_0, a_1, a_2, a_3, a_4, a_5, \dots) = (a_1, a_0, a_3, a_2, a_5, a_4 \dots)$, i.e., given a sequence $\mathbf{a} \in V$, the map U swaps the entries a_{2k} and a_{2k+1} for each $k \in \mathbb{N}$. You are given that U is a vector space isomorphism. Prove that U is self-adjoint using the definition of adjoint map.

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V$. Then we have $\langle U(a_n)_{n\in\mathbb{N}} \mid (b_n)_{n\in\mathbb{N}} \rangle = \langle (a_1, a_0, a_3, a_2, \dots) \mid (b_0, b_1, b_2, b_3, \dots) \rangle = a_1 b_0 + a_0 b_1 + a_3 b_2 + a_2 b_3 + \cdots = a_0 b_1 + a_1 b_0 + a_2 b_3 + a_3 b_2 + \cdots = \langle (a_0, a_1, a_2, a_3, \dots) \mid (b_1, b_0, b_3, b_2, \dots) \rangle = \langle (a_n)_{n\in\mathbb{N}} \mid U(b_n)_{n\in\mathbb{N}} \rangle$. Observe that the third equality holds because the terms being eventually zero implies that we can rearrange the terms of this sum as shown. It follows from the definition of adjoint that $U = U^*$.