PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 90 MINUTES
2 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(6 × 15 pts) **1.** Consider the first quadrant $V = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ in the plane together with the operations $\oplus : V \times V \to V$ and $\otimes : \mathbb{R} \times V \to V$ given by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2,\ y_1y_2) \quad \text{and} \quad \alpha \otimes (x, y) = (x^\alpha,\ y^\alpha)$$

Recall that $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$ **whose zero vector is the vector** $(1, 1)$. Moreover, the set $\mathcal{B} = \{(e, 1), (1, e)\}$ is a basis of V. Indeed, for any $(x, y) \in V$, we have that $(x, y) = \ln(x) \otimes (e, 1)\ \oplus\ \ln(y) \otimes (1, e)$.

(a) Show that the map $f : V \to \mathbb{R}$ defined by $f(x, y) = \ln(x^2 y)$ is a linear functional.

Let $\alpha \in \mathbb{R}$ and $(x, y), (z, w) \in V$. Then we have

$$\begin{aligned} f(\alpha \otimes (x, y) \oplus (z, w)) = f((x^\alpha, y^\alpha) \oplus (z, w)) = f(x^\alpha z, y^\alpha w) &= \ln\left((x^\alpha z)^2 y^\alpha w\right) \\ &= \ln\left((x^2 y)^\alpha z^2 w\right) = \alpha \ln\left(x^2 y\right) + \ln\left(z^2 w\right) \\ &= \alpha f(x, y) + f(z, w) \end{aligned}$$

Hence f is a linear functional.

(b) Explicitly write the elements of the dual basis $\mathcal{B}^*$. Then express the linear functional f in Part (a) as a linear combination of the elements of $\mathcal{B}^*$. By the given information in the question statement, the maps $f_1 : V \to \mathbb{R}$ and $f_2 : V \to \mathbb{R}$ given by $f_1(x, y) = \ln(x)$ and $f_2(x, y) = \ln(y)$ are the coordinate functionals associated to the basis $\mathcal{B}$. Therefore, the set $\mathcal{B}^* = \{f_1, f_2\}$ is the dual basis of $\mathcal{B}$. Moreover, we have $f(x, y) = \ln(x^2 y) = 2\ln(x) + \ln(y) = 2f_1(x, y) + f_2(x, y)$ for all $(x, y) \in V$ and hence $f = 2f_1 + f_2$.

(c) Determine the nullity of f and find a basis for the subspace $\ker(f)$. Since f is a non-zero functional, its image is a non-zero subspace of $\mathbb{R}$ and hence equals $\mathbb{R}$, from which it follows that $\text{rank}(f) = 1$. Since we have $\text{rank}(f) + \text{nullity}(f) = \dim_{\mathbb{R}}(V) = 2$ by the rank-nullity theorem, we obtain that $\text{nullity}(f) = 1$. Since $\dim_{\mathbb{R}}(\ker(f)) = 1$, any non-zero vector in $\ker(f)$ will form a basis for $\ker(f)$. In particular, we have $\left(\frac{1}{2}, 4\right) \in \ker(f)$ as $f\left(\frac{1}{2}, 4\right) = \ln\left(\frac{1}{4} \cdot 4\right) = 0$, and so $\left\{\left(\frac{1}{2}, 4\right)\right\}$ is a basis for $\ker(f)$. **NOTE: This is only *a* solution. You can actually solve this part by explicitly computing $\ker(f)$ as well.**

(d) You are given that the map $U : V \to V$ given by $U(x,y) = \left(xy\ ,\ \dfrac{x^2}{y}\right)$ is a linear transformation. Find the matrix representation $[U]_{\mathcal{B}}$ of the linear transformation U with respect to the ordered basis $\mathcal{B}$. It suffices to find $[U(e,1)]_{\mathcal{B}}$ and $[U(1,e)]_{\mathcal{B}}$ and form a matrix whose columns are these coordinate matrices. Recall that we are given in the question statement that $(x,y) = \ln(x)\otimes(e,1)\ \oplus\ \ln(y)\otimes(1,e)$ for all $(x,y) \in V$. It follows that

$$[U(e,1)]_{\mathcal{B}} = \left[(e,e^2)\right]_{\mathcal{B}} = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } [U(1,e)]_{\mathcal{B}} = \left[(e,e^{-1})\right]_{\mathcal{B}} = \begin{bmatrix}1\\-1\end{bmatrix}$$

Consquently, by a theorem proven in class, we have $[U]_{\mathcal{B}} = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$.

(e) If it exists, find the inverse linear transformation $U^{-1} : V \to V$. If it does not exist, explain why the inverse linear transformation does not exist. Observe that $(x,y) \in \ker(U)$ iff $\left(xy, \frac{x^2}{y}\right) = (1,1)$ iff $x = 1$ and $y = 1$. It follows that $\ker(U) = \{(1,1)\} = \{\mathbf{0}_V\}$ and hence U is injective. Since V is finite dimensional, U must be also surjective. Thus $U^{-1} : V \to V$ exists. In order to find U^{-1}, we shall first find $[U]_{\mathcal{B}}^{-1}$. Applying Gauss-Jordan elimination, we obtain that

$$\left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 2 & -1 & 0 & 1\end{array}\right) \to \left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & -3 & -2 & 1\end{array}\right) \to \left(\begin{array}{cc|cc}1 & 0 & 1/3 & 1/3\\ 0 & -3 & -2 & 1\end{array}\right) \to \left(\begin{array}{cc|cc}1 & 0 & 1/3 & 1/3\\ 0 & 1 & 2/3 & -1/3\end{array}\right)$$

So we have that $[U^{-1}]_{\mathcal{B}} = [U]_{\mathcal{B}}^{-1} = \begin{bmatrix}1/3 & 1/3\\ 2/3 & -1/3\end{bmatrix}$. Since $[(x,y)]_{\mathcal{B}} = \begin{bmatrix}\ln(x)\\ \ln(y)\end{bmatrix}$, we have $[U^{-1}(x,y)]_{\mathcal{B}} = \begin{bmatrix}1/3 & 1/3\\ 2/3 & -1/3\end{bmatrix} \cdot \begin{bmatrix}\ln(x)\\ \ln(y)\end{bmatrix} = \begin{bmatrix}\ln(x^{1/3}y^{1/3})\\ \ln(x^{2/3}y^{-1/3})\end{bmatrix}$. Therefore, for any $(x,y) \in V$, we have $U^{-1}(x,y) = \ln(x^{1/3}y^{1/3}) \otimes (e,1) \oplus \ln(x^{2/3}y^{-1/3}) \oplus (1,e) = \left(\sqrt[3]{xy}, \sqrt[3]{x^2/y}\right)$.

NOTE: This part can be solved without finding the matrix representation of the inverse transformation, by letting $a = xy,\ b = x^2/y$ and finding x, y in terms of a, b.

(f) Let $U^{\mathrm{t}} : V^* \to V^*$ be the transpose of the linear transformation U. Find $\left[(U^{\mathrm{t}})^2\right]_{\mathcal{B}^*}$ where $\mathcal{B}^*$ is the dual basis of $\mathcal{B}$.

$$\left[(U^{\mathrm{t}})^2\right]_{\mathcal{B}^*} = [(U^{\mathrm{t}})]_{\mathcal{B}^*}^2 = \left([U]_{\mathcal{B}}^{\mathrm{T}}\right)^2 = \begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}^2 = \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

(10 pts) **2.** Find an example of a vector space W over the field $\mathbb{R}$ and a **non-zero** linear transformation $T : W \to W$ such that $\ker(T)$ is isomorphic to W.

Consider $W = \mathbb{R}[x]$ and $T : W \to W$ given by $T(f) = f(0)$ for all $f \in W$. Then we have that $\ker(T) = \{f \in \mathbb{R}[x] :\ f(0) = 0\} = \{xg(x) :\ g \in \mathbb{R}[x]\}$, that is, $\ker(T)$ is the subspace of polynomials with constant term 0. Observe that the set $\{x^n : n \geq 1\} = \{x, x^2, x^3, \dots\}$ is a basis for the subspace $\ker(T)$ and the set $\{x^n : n \geq 0\} = \{1, x, x^2, \dots\}$ is a basis for W. Since the bases $\{x^n : n \geq 1\}$ and $\{x^n : n \geq 0\}$ have the same cardinality, we have that $\ker(T)$ and W are isomorphic as vector spaces.