METU Department of Mathematics, Math 261, Midterm II, November 18, 2024

PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 80 MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

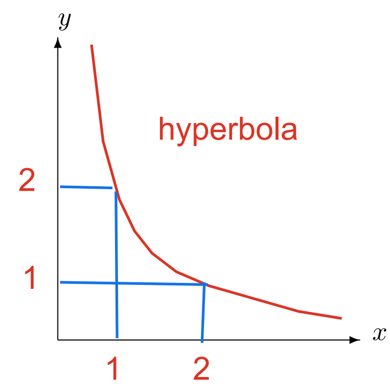
(20+25+25 pts) **1.** Consider the first quadrant $V = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ in the plane together with the operations $\oplus : V \times V \to V$ and $\otimes : \mathbb{R} \times V \to V$ given by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2,\ y_1y_2) \quad \text{and} \quad \alpha \otimes (x, y) = (x^\alpha,\ y^\alpha)$$

Recall that $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$ **whose zero vector is the vector** $(1, 1)$. Moreover, recall that $W = \{(x, y) \in V :\ xy = 1\}$ is a subspace of V.

(a) Find the coset $(1, 2) \oplus W$: First find an equation that defines $(1, 2) \oplus W$ and then sketch it in the given plane on the right.

$(1, 2) \oplus W = \{(1, 2) \oplus (a, b) : (a, b) \in W\} =$
$\{(a, 2b) \in V : (a, b) \in W\} =$
$\{(a, 2b) \in V : ab = 1\} =$
$\{(x, y) \in V : xy = 2\}$ and hence the given coset is defined by the equation $xy = 2$ in the first quadrant.



(b) Show that $\mathcal{B}_1 = \{(1, 2), (3, 4)\}$ is a basis for V.

We first argue that $\mathcal{B}_1$ is linearly independent.
Let $\alpha, \beta \in \mathbb{R}$. Suppose that $\big(\alpha \otimes (1, 2)\big) \oplus \big(\beta \otimes (3, 4)\big) = (1, 1)$. Then, by the definition of operations, we have $(1^\alpha, 2^\alpha) \oplus (3^\beta, 4^\beta) = (3^\beta, 2^\alpha 4^\beta) = (1, 1)$. Since $3^\beta = 1$, we obtain $\beta = 0$. But then we get $2^\alpha = 1$, which implies $\alpha = 0$ as well. Hence $\mathcal{B}_1$ is linearly independent.

We now argue that $\operatorname{span}(\mathcal{B}_1) = V$. Clearly $\operatorname{span}(\mathcal{B}_1) \subseteq V$. Now let $(x, y) \in V$. Consider $\alpha = \log_2(y) - 2\log_3(x)$ and $\beta = \log_3(x)$. Then

$$\big(\alpha \otimes (1, 2)\big) \oplus \big(\beta \otimes (3, 4)\big) = (1^\alpha, 2^\alpha) \oplus (3^\beta, 4^\beta) = (3^\beta, 2^\alpha 4^\beta) = (3^\beta, 2^{\alpha + 2\beta})$$

$$= (3^{\log_3(x)}, 2^{\log_2(y) - 2\log_3(x) + 2\log_3(x)}) = (3^{\log_3(x)}, 2^{\log_2(y)}) = (x, y)$$

Hence $(x, y) \in \operatorname{span}(\mathcal{B}_1)$. This shows that $\operatorname{span}(\mathcal{B}_1) \supseteq V$ and hence $\operatorname{span}(\mathcal{B}_1) = V$.

(c) You are given that $\mathcal{B}_2 = \{(2,1),(1,2)\}$ is also a basis for V. Consider $\mathcal{B}_1$ and $\mathcal{B}_2$ as ordered bases. Find the change of base matrix $P \in M_{2\times 2}(\mathbb{R})$ such that $[v]_{\mathcal{B}_1} = P[v]_{\mathcal{B}_2}$ for all $v \in V$.

To find the desired change of base matrix, we need to find $[(2,1)]_{\mathcal{B}_1}$ and $[(1,2)]_{\mathcal{B}_1}$. In Part (b), we have already found how to write an arbitrary element (x,y) as a linear combination of elements of $\mathcal{B}_1$. Using that argument, we obtain that

$$[(2,1)]_{\mathcal{B}_1} = \begin{bmatrix} \log_2(1) - 2\log_3(2) \\ \log_3(2) \end{bmatrix} = \begin{bmatrix} -2\log_3(2) \\ \log_3(2) \end{bmatrix} \text{ and } [(1,2)]_{\mathcal{B}_1} = \begin{bmatrix} \log_2(2) - 2\log_3(1) \\ \log_3(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, by a theorem proven in class, the matrix

$$P = \begin{bmatrix} -2\log_3(2) & 1 \\ \log_3(2) & 0 \end{bmatrix}$$

is the desired change of base matrix.

(10 pts) **2.** Let V be a vector space over $\mathbb{R}$ with a basis $\mathcal{B}$. Suppose that the set $\mathcal{B}+\mathcal{B} = \{b_1 + b_2 : \ b_1, b_2 \in \mathcal{B}\}$ is also a basis for V. Prove that $\dim_{\mathbb{R}}(V)$ is equal to either 0 or 1.

Assume towards a contradiction that $|\mathcal{B}| = \dim_{\mathbb{R}}(V) \geq 2$. Then there exist distinct elements $a, b \in \mathcal{B}$. Set $u = a + a$, $v = a + b$ and $w = b + b$. Then u, v, w are distinct elements of V and moreover, by the definition of $\mathcal{B}+\mathcal{B}$, we have $u, v, w \in \mathcal{B}+\mathcal{B}$. But then, we obtain

$$\frac{1}{2}u + (-1)v + \frac{1}{2}w = \frac{1}{2}(2a) + (-1)(a+b) + \frac{1}{2}(2b) = 0$$

This implies that $\mathcal{B}+\mathcal{B}$ is linearly dependent, which is a contradiction. Therefore $\dim_{\mathbb{R}}(V) < 2$.

(20 pts) **3.** Find the row rank, i.e. the dimension of the row space, of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ 2 & 4 & 2 & 6 \end{bmatrix}$$

by exhibiting a basis for the row space of A. It suffices to find the row reduced echelon form of A. Applying Gaussian elimination, we get

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ 2 & 4 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

By a theorem proven in class, the set $\{[1\ 2\ 0\ 0], [0\ 0\ 1\ 0], [0\ 0\ 0\ 1]\}$ of non-zero rows of R is a basis for the row space of A. Hence $\text{rowrank}(A) = 3$.