PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 80 MINUTES
2 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(20+20 pts) **1.** Consider the first quadrant $V = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ in the plane together with the operations $\oplus : V \times V \to V$ and $\otimes : \mathbb{R} \times V \to V$ given by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2,\ y_1y_2) \quad \text{and} \quad \alpha \otimes (x, y) = (x^\alpha,\ y^\alpha)$$

(a) You are given that $(V, \oplus)$ is an abelian group with the identity element $\mathbf{1}_V = (1, 1)$. Show that $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$.

We shall check the remaining four vector space properties. For all $(x_1, y_1), (x_2, y_2) \in V$ and for all $\alpha, \beta \in \mathbb{R}$, we have

- $1 \otimes (x_1, y_1) = (x_1^1, y_1^1) = (x_1, y_1)$,
- $\alpha \otimes (\beta \otimes (x_1, y_1)) = \alpha \otimes \left(x_1^\beta, y_1^\beta\right) = \left(\left(x_1^\beta\right)^\alpha, \left(y_1^\beta\right)^\alpha\right) = \left(x_1^{\beta\alpha}, y_1^{\beta\alpha}\right) = \left(x_1^{\alpha\beta}, y_1^{\alpha\beta}\right) = (\alpha\beta) \otimes (x_1, y_1)$,
- $(\alpha + \beta) \otimes (x_1, y_1) = \left(x_1^{\alpha+\beta}, y_1^{\alpha+\beta}\right) = \left(x_1^\alpha x_1^\beta, y_1^\alpha y_1^\beta\right) = (x_1^\alpha, y_1^\alpha) \oplus \left(x_1^\beta, y_1^\beta\right) = (\alpha \otimes (x_1, y_1)) \oplus (\beta \otimes (x_1, y_1))$,
- $\alpha \otimes ((x_1, y_1) \oplus (x_2, y_2)) = \alpha \otimes (x_1x_2, y_1y_2) = ((x_1x_2)^\alpha, (y_1y_2)^\alpha) = (x_1^\alpha x_2^\alpha, y_1^\alpha y_2^\alpha) = (x_1^\alpha, y_1^\alpha) \oplus (x_2^\alpha, y_2^\alpha) = (\alpha \otimes (x_1, y_1)) \oplus (\alpha \otimes (x_2, y_2))$

It follows that $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$.

(b) Determine whether or not the set $W = \{(x, y) \in V : \ xy = 1\}$ is a subspace of V.

First, observe that $\mathbf{1}_V = (1, 1) \in W$ as $1 \cdot 1 = 1$ and so $W \neq \emptyset$. Let $\alpha \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in W$. Consider

$$\alpha \otimes (x_1, y_1) \oplus (x_2, y_2) = (x_1^\alpha, y_1^\alpha) \oplus (x_2, y_2) = (x_1^\alpha x_2, y_1^\alpha y_2)$$

Since $(x_1, y_1), (x_2, y_2) \in W$, we have $x_1y_1 = 1$ and $x_2y_2 = 1$ and consequently $(x_1^\alpha x_2)(y_1^\alpha y_2) = x_1^\alpha y_1^\alpha x_2y_2 = 1$. Hence $\alpha \otimes (x_1, y_1) \oplus (x_2, y_2) \in W$ and thus, W is a subspace of V.

(20+20+20 pts) **2.** Let $m, n \in \mathbb{Z}^+$. In this question, consider $V = \mathrm{M}_{1\times n}(\mathbb{R})$ as a vector space over $\mathbb{R}$. For each $A \in \mathrm{M}_{m\times n}(\mathbb{R})$, define the row space S_A to be the subspace of V generated by the rows of A, that is, $S_A = \langle A_1, \dots, A_m \rangle$ where $A_1, \dots, A_m \in V$ are the rows of A.

(a) Let $A, B \in \mathrm{M}_{m\times n}(\mathbb{R})$. Prove that if B is obtained from A via a single application of an elementary row operation, then $S_B \subseteq S_A$.

Suppose that B is obtained from A via a single application of an elementary row operation. We shall show that the rows of B are contained in S_A. We know split into three cases:

- If the row operation interchanges the i-th row and the j-th row of A, then the set of rows of B are $\{A_1, \dots, A_m\}$,
- If the operation multiplies the i-th row of A by a non-zero $\alpha \in \mathbb{R}$, then the set of rows of B are $\{A_1, \dots, A_{i-1}, \alpha A_i, A_{i+1}, \dots, A_m\}$,
- If the operation multiplies the i-th row of A by $\alpha \in \mathbb{R}$ and adds it to the j-th row then the rows of B are $\{A_1, \dots, A_{j-1}, \alpha A_i + A_j, A_{j+1}, \dots, A_m\}$

Since $\{A_1, \dots, A_m\} \subseteq S_A$ and S_A being a subspace implies that $\alpha X + Y \in S_A$ for any $X, Y \in S_A$, we obtain that, in all cases, the rows of B are contained S_B. Therefore, $S_B \subseteq S_A$ by definition, as S_B is the smallest subspace of V containing the rows of B.

(b) Let $A \in \mathrm{M}_{n\times n}(\mathbb{R})$. Prove that if A is invertible, then $S_A = V$.

Suppose that A is invertible. Then the row reduced echelon form of A is $I_{n\times n}$ and so there exists a sequence of matrices $A = A_0 \to A_1 \to \cdots \to A_k = I_{n\times n}$ each of which is obtained from the previous one via a single elementary row operation. By Part (a), we know that $S_A = S_{A_0} \supseteq S_{A_1} \supseteq \cdots \supseteq S_{A_k} = S_{I_{n\times n}}$. We claim that $S_{I_{n\times n}} = V$. Trivially, we have $S_{I_{n\times n}} \subseteq V$. For the other inclusion, for each $1 \le i \le n$, let R_i denote the i-th row of $I_{n\times n}$. Then, for any $X = \begin{bmatrix} x_{11} & \dots & x_{1n} \end{bmatrix} \in V$, we have $X = \sum_{i=1}^n x_{1i} R_i$ and hence $X \in \langle R_1, \dots, R_n \rangle = S_{I_{n\times n}}$. It follows that $S_{I_{n\times n}} = V$ and so $S_A \supseteq V$. Since we also have $S_A \subseteq V$, we obtain that $S_A = V$.

(c) You are given that the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 1 \\ 2 & 6 & 2 \end{bmatrix}$ is invertible. Therefore, by Part (b), $S_A = \mathrm{M}_{1\times 3}(\mathbb{R})$. Express $\begin{bmatrix} 4 & 13 & -1 \end{bmatrix} \in V$ as a linear combination of the rows of A.

We wish to find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha \cdot \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} + \beta \cdot \begin{bmatrix} 2 & 6 & 1 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 2 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 13 & -1 \end{bmatrix}$. Writing the relevant non-homogeneous system of equations and applying Gaussian elimination, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 2 & 6 & 6 & 13 \\ 4 & 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 2 & 2 & 5 \\ 0 & -7 & -6 & -17 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

Hence, choosing $\alpha = -1$, $\beta = 2$, $\gamma = 1/2$, the initial equation is satisfied.