PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 150 MINUTES
4 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

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***(10 × 6 pts)* 1.** Consider the set $\mathbb{R}^{\mathbb{R}} = \{f \mid \ f : \mathbb{R} \to \mathbb{R} \text{ is a function}\}$ of all functions on $\mathbb{R}$. Then $\mathbb{R}^{\mathbb{R}}$ is a vector space over $\mathbb{R}$, where the vector addition $f+g$ and the scalar multiplication $\alpha \cdot f$ are defined as follows.

$$(f+g)(x) = f(x) + g(x) \text{ and } (\alpha \cdot f)(x) = \alpha f(x) \text{ for every } x \in \mathbb{R}$$

for every $\alpha \in \mathbb{R}$ and $f, g \in \mathbb{R}^{\mathbb{R}}$.

(a) Determine whether or not $W = \{f \in \mathbb{R}^{\mathbb{R}} \mid \ \exists x \in \mathbb{R} \ f(x) = 0\}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Set $f(x) = |x|$ and $g(x) = |x-1|$. Then we have $f, g \in W$, however, $f + g \notin W$ since $(f+g)(x) = |x| + |x-1| \neq 0$ for any $x \in \mathbb{R}$. Therefore, W is not closed under vector addition and hence is not a subspace.

Set $S = \{\sin(x), \cos(x), e^{2x}\} \subseteq \mathbb{R}^{\mathbb{R}}$. In Part (b),(c), (d) and (e), we shall together show that S is linearly independent.

(b) Complete the following statement appropriately: S is linearly independent iff for every $\alpha, \beta, \gamma \in \mathbb{R}$, we have

$$\alpha \cdot \sin(x) + \beta \cdot \cos(x) + \gamma \cdot e^{2x} = 0 \text{ for every } x \in \mathbb{R} \text{ implies that } \alpha = \beta = \gamma = 0$$

(c) In the equation that you wrote in Part (a), by plugging in $x = -\pi, 0, \pi/2$ respectively, you shall obtain a linear system consisting of three equations in variables α, β and γ. Complete the entries of the following matrix A so that it is the coefficient matrix of this linear system.

$$A = \begin{pmatrix} 0 & -1 & e^{-2\pi} \\ 0 & 1 & 1 \\ 1 & 0 & e^{\pi} \end{pmatrix}$$

(d) Find $\det(A)$.

Applying the column expansion formula on the first column, we get

$$\det(A) = (-1)^{1+1} \cdot 0 \cdot \det \begin{pmatrix} 1 & 1 \\ 0 & e^{\pi} \end{pmatrix} + (-1)^{2+1} \cdot 0 \cdot \det \begin{pmatrix} -1 & e^{-2\pi} \\ 0 & e^{\pi} \end{pmatrix} + (-1)^{3+1} \cdot 1 \cdot \det \begin{pmatrix} -1 & e^{-2\pi} \\ 1 & 1 \end{pmatrix} = -1 - e^{-2\pi}$$

(e) Without applying Gaussian elimination, argue that the linear system $AX = Y$ in Part (c) has the unique solution $\alpha = \beta = \gamma = 0$.
Observe that the column matrix Y is the zero column matrix because the constant term in the equation in Part (b) is 0. Since $\det(A) \neq 0$, we have that A is invertible and consequently, the linear system $AX = \mathbf{0}$ has only the trivial solution $\alpha = \beta = \gamma = 0$ as $X = A^{-1}\mathbf{0} = \mathbf{0}$.
Having shown that S is linearly independent, consider the subspace $V = \langle S \rangle$ spanned by the set S. Then $S = \{\sin(x), \cos(x), e^{2x}\}$ is an ordered basis for V. In Part (f), (g), (h), (i) and (j), consider V as a separate vector space on its own.
(f) You are given that the differentiation map $D : V \to V$ given by $D(f) = f'$ is a linear transformation. Find its matrix $[D]_S$ relative to the ordered basis S.

Observe that $[D(\sin(x))]_S = [\cos(x)]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $[D(\cos(x))]_S = [-\sin(x)]_S = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ and $[D(e^{2x})]_S = [2e^{2x}]_S = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Hence we have $[D]_S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(g) Let $U : V \to V$ be a linear transformation such that $[U]_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Find $U(2\sin(x) + 6\cos(x) + e^{2x})$.

$$[U(2\sin(x)+6\cos(x)+e^{2x})]_S = [U]_S \cdot [2\sin(x)+6\cos(x)+e^{2x}]_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ 1 \end{bmatrix}$$

and so $U(2\sin(x) + 6\cos(x) + e^{2x}) = 2\sin(x) + 9\cos(x) + e^{2x}$.

Recall that every element of V is of the form $\alpha \cdot \sin(x) + \beta \cdot \cos(x) + \gamma \cdot e^{2x}$.
(h) Consider the linear functional $\varphi : V \to \mathbb{R}$ given by $\varphi(f) = f(0)$. Describe the subspace $\ker(\varphi)$ in the set-builder notation using an equality only involving α, β or γ.

$$\ker(\varphi) = \left\{ \alpha \cdot \sin(x) + \beta \cdot \cos(x) + \gamma \cdot e^{2x} \in V \mid \beta + \gamma = 0 \right\}$$

(i) Find a basis for $\ker(\varphi)$. Since φ is a non-zero functional, its image is a non-zero subspace of $\mathbb{R}$ and hence equals $\mathbb{R}$, from which it follows that $\operatorname{rank}(\varphi) = 1$. Since we have $\operatorname{rank}(\varphi) + \operatorname{nullity}(\varphi) = \dim_{\mathbb{R}}(V) = 3$ by the rank-nullity theorem, we obtain that $\operatorname{nullity}(\varphi) = 2$. Therefore, any subset of $\ker(\varphi)$ that consists of two linearly independent vectors will form a basis for $\ker(\varphi)$. Consider $\mathcal{B} = \{\sin(x), \cos(x) - e^{2x}\}$. Then $\mathcal{B} \subseteq \ker(\varphi)$ because $\varphi(\sin(x)) = \sin(0) = 0$ and $\varphi(\cos(x) - e^{2x}) = \cos(0) - e^0 = 0$. Moreover, if $\mathcal{B}$ were linearly dependent, then S would be linearly dependent. Thus $\mathcal{B}$ is linearly independent and consequently, is a basis for $\ker(\varphi)$.

For the next part of the question, consider $\mathbb{R}^3$ as a vector space over $\mathbb{R}$ together with its standard operations.

(j) Find an isomorphism $T : V \to \mathbb{R}^3$ such that the image of $\ker(\varphi)$ under T is equal to the xy-plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and $T(e^{2x}) = (0, 0, 1)$.

$$T\left(\alpha \cdot \sin(x) + \beta \cdot \cos(x) + \gamma \cdot e^{2x}\right) = (\alpha, \beta, \beta + \gamma)$$

For this part of the question, you need not check that the map T that you define is a linear transformation, however, you need to check that it has the other desired properties.
Let T be defined as above. Then, by the definition of T, we have that $T(\sin(x)) = (1, 0, 0)$ and $T(\cos(x) - e^{2x}) = (0, 1, 0)$ and $T(e^{2x}) = (0, 0, 1)$. Moreover the subspace

$$\langle (1,0,0), (0,1,0) \rangle = \langle T(\sin(x)), T(\cos(x) - e^{2x}) \rangle = T\Big(\langle \sin(x), \cos(x) - e^{2x} \rangle\Big) = T\Big(\ker(\varphi)\Big)$$

equals the xy-plane.
Observe that $T(\alpha \cdot \sin(x) + \beta \cdot \cos(x) + \gamma \cdot e^{2x}) = (\alpha, \beta, \beta + \gamma) = (0, 0, 0)$ iff $\alpha = \beta = \gamma = 0$ and hence $\ker(T) = \{\mathbf{0}\}$ is the trivial subspace consisting of the zero function. It follows that T is injective. However, since $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \mathbb{R}^3 = 3$ and T is injective, T is also surjective. Therefore, T is an isomorphism.

***(10 pts)* 2.** Let $n \geq 2$ be an integer and $A \in M_{n\times n}(\mathbb{Z})$. Suppose that each entry of A is an odd integer. Show that $\det(A)$ is an even integer.

Recall that $\det(A) = \sum_{\varphi \in S_n} \operatorname{sgn}(\varphi) \prod_{i=1}^{n} A_{i\varphi(i)}$. Since each entry of A is an odd integer, the term $\operatorname{sgn}(\varphi) \prod_{i=1}^{n} A_{i\varphi(i)}$ is odd for every $\varphi \in S_n$ being a product of odd integers. But then, since $|S_n| = n!$ is even, being a sum of even number many odd integers, $\det(A)$ is even.
Alternatively, this fact can be proven by induction on n using the column/row expansion formula of determinant.

***(6+6+6 pts)* 3.** In this question, we shall work over the field $\mathbb{F} = \mathbb{Z}_3 = \{0, 1, 2\}$ with three elements. Consider the matrix $M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ over the field $\mathbb{Z}_3$.

(a) If exists, find the inverse M^{-1} using Gaussian elimination.

$$\left(\begin{array}{cccc|cccc} 1&1&0&1&1&0&0&0 \\ 0&1&0&1&0&1&0&0 \\ 0&0&2&0&0&0&1&0 \\ 0&0&1&1&0&0&0&1 \end{array}\right) \xrightarrow[2R_2+R_1\to R_1]{} \left(\begin{array}{cccc|cccc} 1&0&0&0&1&2&0&0 \\ 0&1&0&1&0&1&0&0 \\ 0&0&2&0&0&0&1&0 \\ 0&0&1&1&0&0&0&1 \end{array}\right) \xrightarrow[R_3+R_4\to R_4]{} \left(\begin{array}{cccc|cccc} 1&0&0&0&1&2&0&0 \\ 0&1&0&1&0&1&0&0 \\ 0&0&2&0&0&0&1&0 \\ 0&0&0&1&0&0&1&1 \end{array}\right) \xrightarrow[2R_4+R_2\to R_2]{}$$

$$\left(\begin{array}{cccc|cccc} 1&0&0&0&1&2&0&0 \\ 0&1&0&0&0&1&2&2 \\ 0&0&2&0&0&0&1&0 \\ 0&0&0&1&0&0&1&1 \end{array}\right) \xrightarrow[2R_3\to R_3]{} \left(\begin{array}{cccc|cccc} 1&0&0&0&1&2&0&0 \\ 0&1&0&0&0&1&2&2 \\ 0&0&1&0&0&0&2&0 \\ 0&0&0&1&0&0&1&1 \end{array}\right) \text{ and so } M^{-1} = \begin{bmatrix} 1&2&0&0 \\ 0&1&2&2 \\ 0&0&2&0 \\ 0&0&1&1 \end{bmatrix}$$

(b) Find the determinant $\det(M)$.

Observe that, in the row reduction process above, we only applied the row operation of "multiplying a row by a constant and adding it to another row" until we obtained the matrix

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since this row operation does not change the determinant, we have that $\det(M) = \det(N) = 1 \cdot 1 \cdot 2 \cdot 1 = 2$. Alternatively, one may find $\det(M)$ using a row/column expansion formula.

(c) Find the adjoint matrix $\mathrm{adj}(M)$ without computing any determinants except $\det(M)$.

Since $M^{-1} = (\det(M))^{-1}\mathrm{adj}(M)$, we have that

$$\mathrm{adj}(M) = \det(M)M^{-1} = 2\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

***(6+6 pts)* 4.** For the parts of this question, which were asked as midterm questions during the semester, *no partial credit will be given unless the answer is completely correct.*

(a) Let V be a vector space over $\mathbb{R}$ with a basis $\mathcal{B}$. Suppose that the set

$$\mathcal{B} + \mathcal{B} = \{b_1 + b_2 : \ b_1, b_2 \in \mathcal{B}\}$$

is also a basis for V. Prove that $\dim_{\mathbb{R}}(V)$ is equal to either 0 or 1.

This question was already asked in a past midterm and its solution was already posted.

(b) Find an example of a vector space W over the field $\mathbb{R}$ and a **non-zero** linear transformation $T : W \to W$ such that $\ker(T)$ is isomorphic to W. For this part of the question, it is sufficient to define W and $T : W \to W$ explicitly without proving that they are as desired.

This question was already asked in a past midterm and its solution was already posted.