M E T U   Department of Mathematics

MATH 124 2023-2024 Academic Year Spring Semester Midterm II April 22, 2024, 17:40		
F U L L N A M E	S T U D E N T I D	DURATION 80 MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

***(15+20+15 pts)* 1.**

a) Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$ is *not* a subspace of the vector space $\mathbb{R}^3$.

Observe that $(1, 1, 0) \in S$ and $(0, 0, 1) \in S$ but $(1, 1, 0) + (0, 0, 1) = (1, 1, 1) \notin S$. Since a subspace of a vector space is closed with respect to vector addition, we have that S is not a subspace.

b) Show that the vectors $(1, 0, 1)$, $(2, 1, 2)$, $(0, 1, 1)$ in $\mathbb{R}^3$ are linearly independent.

Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha(1, 0, 1) + \beta(2, 1, 2) + \gamma(0, 1, 1) = \vec{\mathbf{0}} = (0, 0, 0)$. We wish to show that $\alpha = \beta = \gamma = 0$. From the previous equality, we must have

$$\begin{aligned} \alpha + 2\beta &= 0 \\ \beta + \gamma &= 0 \\ \alpha + 2\beta + \gamma &= 0 \end{aligned}$$

Applying Gaussian elimination to this system of equations, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \xrightarrow[-R_1+R_3 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[-R_3+R_2 \to R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[-R_2+R_1 \to R_1]{} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Therefore, the system has a unique solution $\alpha = \beta = \gamma = 0$. It follows that $(1, 0, 1)$, $(2, 1, 2)$, $(0, 1, 1)$ are linearly independent.

c) Find a value $r \in \mathbb{R}$ such that the vectors $(1, 0, 1)$, $(2, 1, r)$, $(0, 1, 1)$ in $\mathbb{R}^3$ are linearly dependent. For the value $r \in \mathbb{R}$ that you have found, express the vector $(2, 1, r)$ as a linear combination of $(1, 0, 1)$ and $(0, 1, 1)$.

We wish to find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha(1, 0, 1) + \beta(2, 1, r) + \gamma(0, 1, 1) = \vec{\mathbf{0}} = (0, 0, 0)$ and at least one of α, β, γ is non-zero. As in Part (b), the relevant system of equations and applying elementary row operations give

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & r & 1 & 0 \end{array}\right) \xrightarrow[-R_1+R_3 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & r-2 & 1 & 0 \end{array}\right)$$

If we choose $r = 3$, then we get a zero row after the row operation $-R_2 + R_3 \to R_3$ and hence the relevant system of equations has a non-zero solution, i.e. there exists such α, β, γ at least one of which is non-zero. Indeed, for $r = 3$, we have that $(2, 1, 3) = 2 \cdot (1, 0, 1) + 1 \cdot (0, 1, 1)$.

(10+10 pts) **2.** Determine whether each of the following statements is true of false. If the statement is true, prove it. If the statement is false, provide an *explicit* counterexample.

a) For all **unit** vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$, if $\vec{u} \bullet \vec{v} = \vec{u} \bullet \vec{w}$, then $\vec{v} \bullet \vec{w} = 1$.

This statement is **false**. For a counterexample, choose $\vec{u} = (1,0)$, $\vec{v} = (0,1)$ and $\vec{w} = (0,-1)$. Then we have that $\vec{u} \bullet \vec{v} = 1 \cdot 0 + 0 \cdot 1 = 0 = 1 \cdot 0 + 0 \cdot (-1) = \vec{u} \bullet \vec{w}$, however, $\vec{v} \bullet \vec{w} = 0 \cdot 0 + 1 \cdot (-1) = -1 \neq 1$

b) For all vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, if $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$ for every $\vec{c} \in \mathbb{R}^3$, then $\vec{a} = \vec{b}$.

This statement is **true**. Let us prove it by a direct proof: Let $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. Assume that $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$ for every $\vec{c} \in \mathbb{R}^3$. In particular, $\vec{a} \times (0,0,1) = \vec{b} \times (0,0,1)$ and $\vec{a} \times (1,0,0) = \vec{b} \times (1,0,0)$. Computing the first cross product, we get

$$(a_2, -a_1, 0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 1 \end{vmatrix} = \vec{a} \times (0,0,1) = \vec{b} \times (0,0,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix} = (b_2, -b_1, 0)$$

Hence $a_1 = b_1$ and $a_2 = b_2$. Computing the second cross product, we get

$$(0, a_3, -a_2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix} = \vec{a} \times (1,0,0) = \vec{b} \times (1,0,0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ 1 & 0 & 0 \end{vmatrix} = (0, b_3, -b_2)$$

Hence $a_3 = b_3$ as well. It follows that $\vec{a} = \vec{b}$.

(15+15 pts) **3.** Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ that passes through the points $(3,2,1), (1,2,3), (2,1,3)$.

a) Find an equation for the plane $\mathcal{P}$.

Let $A = (3,2,1), B = (1,2,3), C = (2,1,3)$. Since the plane $\mathcal{P}$ contains the points A, B, C, the vectors $\vec{AB}$ and $\vec{AC}$ are parallel to the plane $\mathcal{P}$. Hence a normal vector the plane $\mathcal{P}$, which is supposed to be perpendicular to both $\vec{AB}$ and $\vec{AC}$, can be given by

$$\vec{AB} \times \vec{AC} = ((1,2,3) - (3,2,1)) \times ((2,1,3) - (3,2,1)) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ -1 & -1 & 2 \end{vmatrix} = (2,2,2)$$

Thus, using the point A and the normal $\vec{n} = (2,2,2)$, an equation of the plane $\mathcal{P}$ can be given by

$$\begin{aligned} \vec{n} \bullet ((x,y,z) - (3,2,1)) &= 0 \\ 2(x-3) + 2(y-2) + 2(z-2) &= 0 \\ x + y + z &= 6 \end{aligned}$$

b) Find a parametric equation of a line ℓ that passes through the point $(1,1,1)$ and that does not intersect the plane $\mathcal{P}$.
Observe that there exist infinitely many such lines and direction vectors of each of these lines are all parallel to $\mathcal{P}$, that is, perpendicular to a normal vector of $\mathcal{P}$. Consequently, in order to find such a line, it suffices to find a direction vector $\vec{u}$ that is perpendicular to a normal vector of $\mathcal{P}$. Hence, it follows from what we found in Part (a) that the line ℓ that passing through $(1,1,1)$ with direction vector $\vec{AB} = (-2,0,2)$ does not intersect $\mathcal{P}$. An equation for ℓ can be given by

$$\ell : x = 1 - 2\lambda, \;\; y = 1, \;\; z = 1 + 2\lambda, \quad \lambda \in \mathbb{R}$$