M E T U Department of Mathematics

MATH 124 2023-2024 Academic Year Spring Semester Midterm I March 11, 2024, 17:40		
F U L L N A M E	S T U D E N T I D	DURATION 90 MINUTES
3 QUESTIONS ON 2 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(10+20+20 pts) **1.** Let $t \in \mathbb{R}$ be a constant. Consider the following system of linear equations

$$\begin{aligned} x + y + 2z &= t \\ 3x + 5y + 4z &= 2t + 1 \\ 4x + 6y + 6z &= 2t - 2 \end{aligned}$$

a) Write the augmented matrix of this system of linear equations: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & t \\ 3 & 5 & 4 & 2t+1 \\ 4 & 6 & 6 & 2t-2 \end{array}\right)$

Recall that a matrix is said to be in *echelon form* if all zero rows are below non-zero rows and the first non-zero entry of each non-zero row is on the right of the first non-zero entries of all the rows above it.

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b) Using Gaussian elimination, find an echelon form of the augmented matrix that you found in Part (a).
Applying elementary row operations, we get

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & t \\ 3 & 5 & 4 & 2t+1 \\ 4 & 6 & 6 & 2t-2 \end{array}\right) \xrightarrow[-3R_1+R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & t \\ 0 & 2 & -2 & -t+1 \\ 4 & 6 & 6 & 2t-2 \end{array}\right) \xrightarrow[-4R_1+R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & t \\ 0 & 2 & -2 & -t+1 \\ 0 & 2 & -2 & -2t-2 \end{array}\right) \xrightarrow[-R_2+R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & t \\ 0 & 2 & -2 & -t+1 \\ 0 & 0 & 0 & -t-3 \end{array}\right)$$

The resulting matrix is in echelon form.

c) Using the matrix in echelon form that you found in Part (b), determine for which values of t this system of linear equations has no solution, has a unique solution and has infinitely many solutions.

Suppose that $t \neq -3$. In this case, the echelon matrix we obtain has a row of the form $\left(\begin{array}{ccc|c} 0 & 0 & 0 & -t-3 \end{array}\right)$ with $-t-3 \neq 0$ which leads to a contradictory system and so the original system has no solution.

Suppose that $t = -3$. Then the obtained echelon matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 2 & -3 \\ 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$ which corresponds to the system $x + y + 2z = -3,\ 2y - 2z = 4$. This system has infinitely many solutions. Indeed, let $c \in \mathbb{R}$. Setting $z = c$, we obtain that $y = c + 2$ and $x = 3c - 5$. Thus $(x, y, z) = (3c - 5, c + 2, c)$ is a solution for every $c \in \mathbb{R}$. Therefore the original has infinitely many solutions in the case that $t = -3$.

There is no value of t for which the original system has a unique solution.

(15+15 pts) **2.** For each of the following matrices, if it exists, find the inverse of the matrix; if it does not exist, explain why it does not exist.

a) $\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$ Applying Gaussian elimination, we get

$$\left(\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right) \xrightarrow[-1/2R_1+R_2]{} \left(\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right) \xrightarrow[-14R_2+R_1]{} \left(\begin{array}{cc|cc} 2 & 0 & 8 & -14 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right)$$

$$\xrightarrow[1/2R_1]{} \left(\begin{array}{cc|cc} 1 & 0 & 4 & -7 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right) \xrightarrow[2R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & 4 & -7 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

Since we obtained the identity matrix on the left of the partitioned matrix, the matrix $\begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}$ is the inverse of the original matrix. **One can alternatively find the inverse directly using the adjoint matrix and the determinant via the formula.**

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 5 \\ 2 & 4 & 5 \end{bmatrix}$ We have $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 5 \\ 2 & 4 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 5 \\ 4 & 5 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 5 \\ 2 & 5 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 4 \end{vmatrix} = -10 - (-10) = 0$. Since the determinant is 0, this matrix is not invertible. **One can alternatively observe that this matrix is not invertible by applying Gaussian elimination and obtaining a zero row in the process.**

(10+10 pts) **3.** Let $M_{2\times 2}(\mathbb{R})$ denote the set of $2 \times 2-$matrices over $\mathbb{R}$. Prove or disprove the following statements.

a) For any $A \in M_{2\times 2}(\mathbb{R})$, if A is a diagonal matrix, then $AB = BA$ for all $B \in M_{2\times 2}(\mathbb{R})$.

This statement is false. We shall give a counterexample. Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then A is a diagonal matrix but, for $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we obtain

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = BA$$

b) For any $A \in M_{2\times 2}(\mathbb{R})$, if $AB = BA$ for all $B \in M_{2\times 2}(\mathbb{R})$, then A is a diagonal matrix.

This statement is true. To prove it, let $A \in M_{2\times 2}(\mathbb{R})$, say, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose that $AB = BA$ for all $B \in M_{2\times 2}(\mathbb{R})$. From the assumption, we know that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Computing both sides, we get $\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$. Hence $c = 0$ and $a = d$.

Similarly, since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get $\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$. Hence $b = 0$ and $a = d$. It follows that $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and hence A is a diagonal matrix, indeed, is of the form aI.