M E T U   Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2025 Midterm II 10 December 2025 17:40		
F U L L N A M E	S T U D E N T I D	DURATION 120 MINUTES
6 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

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In all your answers, you are expected to **explain and justify** your argument, unless specified otherwise.

(10 pts) **1.** Prove or disprove the following: $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$ for every set A and B.

We shall **disprove** this statement by giving a counterexample. Consider the set $A = \{0, 1\}$ and $B = \{0\}$. Then we have

$$\mathcal{P}(A) - \mathcal{P}(B) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} - \{\emptyset, \{0\}\} = \{\{1\}, \{0, 1\}\}$$

and

$$\mathcal{P}(A - B) = \mathcal{P}(\{0, 1\} - \{0\}) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$$

Since $\{0, 1\} \in \mathcal{P}(A) - \mathcal{P}(B)$ but $\{0, 1\} \notin \mathcal{P}(A - B)$, we have that $\mathcal{P}(A) - \mathcal{P}(B) \nsubseteq \mathcal{P}(A - B)$.

(10+10 pts) **2.** Let A, B, C, D be sets.

a) Prove that $(A \cup B) - (C \cup D) \subseteq (A - C) \cup (B - D)$.

Let $x \in (A \cup B) - (C \cup D)$. Then, by definition of set difference, $x \in A \cup B$ and $x \notin C \cup D$. It follows that, $x \in A$ or $x \in B$, and, $x \notin C$ and $x \notin D$. We now split into two cases.

- **Case I ($x \in A$):** In this case, since $x \in A$ and $x \notin C$, we have $x \in A - C$ and consequently, it holds that $x \in A - C$ or $x \in B - D$. By definition, this means $x \in (A - C) \cup (B - D)$.
- **Case II ($x \in B$):** In this case, since $x \in B$ and $x \notin D$, we have $x \in B - D$ and consequently, it holds that $x \in A - C$ or $x \in B - D$. By definition, this means $x \in (A - C) \cup (B - D)$.

In both cases, we obtained $x \in (A - C) \cup (B - D)$. This completes the proof that $(A \cup B) - (C \cup D) \subseteq (A - C) \cup (B - D)$.

b) Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Let $r \in (A \times B) \cup (C \times D)$. Then $r \in A \times B$ or $r \in C \times D$. We split into cases.

- **Case I ($r \in A \times B$):** In this case, by definition of cartesian product, $r = (a, b)$ for some $a \in A$ and $b \in B$. Since $A \subseteq A \cup C$ and $B \subseteq B \cup D$, we have $r = (a, b) \in (A \cup C) \times (B \cup D)$.
- **Case II ($r \in C \times D$):** In this case, by definition of cartesian product, $r = (c, d)$ for some $c \in C$ and $d \in D$. Since $C \subseteq A \cup C$ and $D \subseteq B \cup D$, we have $r = (c, d) \in (A \cup C) \times (B \cup D)$.

In both cases, we obtained that $r \in (C \times D) \subseteq (A \cup C) \times (B \cup D)$. This completes the proof that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Alternative proof: Let $(x, y) \in (A \times B) \cup (C \times D)$. Then, by definition of union, $(x, y) \in A \times B$ or $(x, y) \in C \times D$. We split into cases.

Case I ($(x, y) \in A \times B$): In this case, $x \in A$ and $y \in B$. Since $A \subseteq A \cup C$ and $B \subseteq B \cup D$, it follows that $(x, y) \in (A \cup C) \times (B \cup D)$.

Case II ($(x, y) \in C \times D$): In this case, $x \in C$ and $y \in D$. Since $C \subseteq A \cup C$ and $D \subseteq B \cup D$, it follows that $(x, y) \in (A \cup C) \times (B \cup D)$. This completes the proof that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

(12 pts) **3.** Let A, B be sets and $f : A \to B$ be a function. Consider the function $g : A \times A \to B \times B$ given by

$$g(x, y) = (f(x), f(y))$$

for all $x, y \in A$. Prove that f is bijective if and only if g is bijective.

($\Rightarrow$) For the left-to-right direction, assume that f is bijective. We wish to show that g is bijective.

- Let $x, y, \widehat{x}, \widehat{y} \in A \times A$. Assume that $g(x, y) = g(\widehat{x}, \widehat{y})$. Then we have $(f(x), f(y)) = (f(\widehat{x}), f(\widehat{y}))$ which subsequently implies that $f(x) = f(\widehat{x})$ and $f(y) = f(\widehat{y})$. Since f is bijective, f is injective and hence $x = \widehat{x}$ and $y = \widehat{y}$. Thus $(x, y) = (\widehat{x}, \widehat{y})$, which shows that g is injective.
- Let $(z, w) \in B \times B$. Clearly $z, w \in B$. Since f is bijective, f is surjective and consequently, there exists $x \in A$ and $y \in A$ such that $f(x) = z$ and $f(y) = w$. But then $(x, y) \in A \times A$ and $g(x, y) = (f(x), f(y)) = (z, w)$. This shows that g is surjective.

Therefore, g is bijective.

($\Leftarrow$) For the right-to-left direction, assume that g is bijective. We wish to show that f is bijective.

- Let $x, y \in A$. Assume that $f(x) = f(y)$. Then $g(x, x) = (f(x), f(x)) = (f(y), f(y)) = g(y, y)$. Since g is bijective, g is injective and consequently, $(x, x) = (y, y)$ which implies $x = y$. This shows that f is injective.
- Let $y \in B$. Clearly $(y, y) \in B \times B$. Since g is bijective, g is surjective and consequently, there exists $(x, \widehat{x}) \in A \times A$ such that $g(x, \widehat{x}) = (y, y)$. But then, by the definition of g, we have $(f(x), f(\widehat{x})) = (y, y)$. In particular, $f(x) = y$ where $x \in A$. This shows that f is surjective.

Therefore, f is bijective.

(3+5+5+5+5 pts) **4.** Consider the functions $\varphi : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ and $\sigma : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ given by

$$\varphi(A) = \{-a \mid a \in A\} \text{ and } \sigma(A) = \{a + 1 \mid a \in A\}$$

for all $A \in \mathcal{P}(\mathbb{Z})$. For example, $\varphi(\{-1, 0, 2\}) = \{-2, 0, 1\}$ and $\sigma(\{-1, 0, 2\}) = \{0, 1, 3\}$. You are given that both φ and σ are invertible functions.

a) Find the inverses $\varphi^{-1} : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ and $\sigma^{-1} : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ by explicitly expressing their rules using the set notation as in the examples above. **For this part of this question only**, you need not justify your answer.

$$\varphi^{-1}(A) = \{\ -a \mid a \in A\} \text{ for all } A \in \mathcal{P}(\mathbb{Z})$$

$$\sigma^{-1}(A) = \{\ a - 1 \mid a \in A\} \text{ for all } A \in \mathcal{P}(\mathbb{Z})$$

b) Show that $\varphi \circ \sigma = \sigma^{-1} \circ \varphi$.

Observe that the domains and the codomains of both $\varphi \circ \sigma$ and $\sigma^{-1} \circ \varphi$ are $\mathcal{P}(\mathbb{Z})$. Let $A \in \mathcal{P}(\mathbb{Z})$. Then, by the definitions of these functions above, we have

$$(\varphi\circ\sigma)(A) = \varphi(\sigma(A)) = \varphi(\{a+1 \mid a \in A\}) = \{-x \mid x \in \{a+1 \mid a \in A\}\} = \{-(a+1) \mid a \in A\} = \{-a-1 \mid a \in A\}$$

$$(\sigma^{-1} \circ \varphi)(A) = \sigma^{-1}(\varphi(A)) = \sigma^{-1}(\{-a \mid a \in A\}) = \{x - 1 \mid x \in \{-a \mid a \in A\}\} = \{-a - 1 \mid a \in A\}$$

Since these functions take the same value at every point in their common domain, we have $\varphi\circ\sigma = \sigma^{-1}\circ\varphi$.

c) Find $(\sigma \circ \varphi \circ \sigma \circ \varphi \circ \sigma)(\{111, 113, 123\})$.

By Part (b), we have that $\varphi \circ \sigma = \sigma^{-1} \circ \varphi$. Composing both sides of this equality with σ from left gives $\sigma \circ \varphi \circ \sigma = \sigma \circ \sigma^{-1} \circ \varphi = \varphi$. Recall from Part (a) that $\varphi^{-1} = \varphi$. Now, composing both sides with $\varphi^{-1} = \varphi$ from right gives $\sigma \circ \varphi \circ \sigma \circ \varphi = \varphi \circ \varphi = \mathbf{1}_{\mathcal{P}(\mathbb{Z})}$. It follows that

$$(\sigma \circ \varphi \circ \sigma \circ \varphi \circ \sigma)(\{111, 113, 123\}) = \left(\mathbf{1}_{\mathcal{P}(\mathbb{Z})} \circ \sigma\right)(\{111, 113, 123\}) = \sigma(\{111, 113, 123\}) = \{112, 114, 124\}$$

In Part (c) and (d) of the question, consider $\widehat{\sigma} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{Z})$, the restriction of the function σ to $\mathcal{P}(\mathbb{N})$, that is, $\widehat{\sigma} = \sigma|_{\mathcal{P}(\mathbb{N})}$.

d) Determine whether or not $\widehat{\sigma}$ has a right inverse. If exists, explicitly find a right inverse. If not, explain why a right inverse does not exist.

We claim that $\widehat{\sigma}$ is not surjective. Consider $B = \{0\} \in \mathcal{P}(\mathbb{Z})$. If there were $A \in \mathcal{P}(\mathbb{N})$ with $\widehat{\sigma}(A) = B$, then we would have

$$\{a + 1 \mid a \in A\} = \widehat{\sigma}(A) = \{0\}$$

so there would be $a \in A$ with $a + 1 = 0$, and hence we would have $-1 \in A \subseteq \mathbb{N}$, which is a contradiction. Thus, B is not in the range of $\widehat{\sigma}$ and so, $\widehat{\sigma}$ is nor surjective. Since $\widehat{\sigma}$ is not surjective, $\widehat{\sigma}$ does not have a right inverse.

e) Determine whether or not $\widehat{\sigma}$ has a left inverse. If exists, explicitly find a left inverse. If not, explain why a left inverse does not exist.

We claim that $\widehat{\sigma}$ is injective. Let $A, B \in \mathcal{P}(\mathbb{N})$. Assume that $\widehat{\sigma}(A) = \widehat{\sigma}(B)$. Then, since $\widehat{\sigma}$ is the restriction of σ, we have $\sigma(A) = \sigma(B)$ and it follows from injectivity of σ, which we are given as a part of question, that $A = B$. Hence $\widehat{\sigma}$ is injective and consequently, $\widehat{\sigma}$ has a left inverse. A left inverse for $\widehat{\sigma}$ can be constructed as follows.

Consider $\delta : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{N})$ given by $\delta(A) = \{|a - 1| \mid a \in A\}$ for all $A \in \mathcal{P}(\mathbb{Z})$. We claim that $\delta \circ \widehat{\sigma} = \mathbf{1}_{\mathcal{P}(\mathbb{N})}$. Let $A \in \mathcal{P}(\mathbb{N})$. Then we have

$$(\delta\circ\widehat{\sigma})(A) = \delta(\{a+1 \mid a \in A\}) = \{|b-1| \mid b \in \{a+1 \mid a \in A\}\} = \{|a| \mid a \in A\} = \{a \mid a \in A\} = A = \mathbf{1}_{\mathcal{P}(\mathbb{N})}(A)$$

Note that the third equality from last holds because $A \subseteq \mathbb{N}$. It follows that δ is a left inverse of $\widehat{\sigma}$.

(5+5+5 pts) **5.** Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projections maps given by

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y$$

for all $x, y \in \mathbb{R}$.

a) Show that, for all $A \subseteq \mathbb{R}$, we have $\pi_1^{-1}(A) = A \times \mathbb{R}$ and $\pi_2^{-1}(A) = \mathbb{R} \times A$. Let $A \subseteq \mathbb{R}$.

We shall first prove that $\pi_1^{-1}(A) = A \times \mathbb{R}$. Let $(x, y) \in \pi_1^{-1}(A)$. Then, by definition of inverse image, $x = \pi_1(x, y) \in A$ and hence $(x, y) \in A \times \mathbb{R}$. This proves that $\pi_1^{-1}(A) \subseteq A \times \mathbb{R}$. For the converse inclusion, let $(x, y) \in A \times \mathbb{R}$. Then $\pi_1(x, y) = x \in A$ and hence, by definition of inverse image, $(x, y) \in \pi_1^{-1}(A)$. This proves that $\pi_1^{-1}(A) \supseteq A \times \mathbb{R}$.

We shall next prove that $\pi_2^{-1}(A) = \mathbb{R} \times A$. Let $(x, y) \in \pi_2^{-1}(A)$. Then, by definition of inverse image, $y = \pi_2(x, y) \in A$ and hence $(x, y) \in \mathbb{R} \times A$. This proves that $\pi_2^{-1}(A) \subseteq \mathbb{R} \times A$. For the converse inclusion, let $(x, y) \in \mathbb{R} \times A$. Then $\pi_2(x, y) = y \in A$ and hence, by definition of inverse image, $(x, y) \in \pi_2^{-1}(A)$. This proves that $\pi_2^{-1}(A) \supseteq \mathbb{R} \times A$.

b) Show that, for all $A, B \subseteq \mathbb{R}$, we have $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$. Let $A, B \subseteq \mathbb{R}$.

By a theorem proven in class, $(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = (A \cap \mathbb{R}) \times (\mathbb{R} \cap B) = A \times B$.

(**Alternatively, you can prove this equality yourself:** We shall prove that $A\times B = (A\times\mathbb{R})\cap(\mathbb{R}\times B)$. For the left-to-right inclusion, let $(x, y) \in A\times B$. Then $x \in A$ and $y \in B$, which implies $(x, y) \in A\times\mathbb{R}$ and $(x, y) \in \mathbb{R}\times B$, and so, $(x, y) \in (A\times\mathbb{R})\cap(\mathbb{R}\times B)$. For the right-to-left inclusion, let $(x, y) \in (A\times\mathbb{R})\cap(\mathbb{R}\times B)$. Then, $(x, y) \in A\times\mathbb{R}$ and $(x, y) \in \mathbb{R}\times B$. These imply $x \in A$ and $y \in B$ respectively. Thus $(x, y) \in A\times B$.)

Now, by Part (a), we have that $A \times B = (A \times \mathbb{R}) \cap (\mathbb{R} \times B) = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$.

c) Prove or disprove the following: For all $P \subseteq \mathbb{R} \times \mathbb{R}$, we have $P = \pi_1(P) \times \pi_2(P)$.
We shall **disprove** this statement by giving a counterexample. Consider $P\{(0,0),(1,1)\} \subseteq \mathbb{R} \times \mathbb{R}$. Then $\pi_1(P) = \{\pi_1(0,0), \pi_1(1,1)\} = \{0,1\}$ and $\pi_2(P) = \{\pi_2(0,0), \pi_2(1,1)\} = \{0,1\}$. It follows that

$$\pi_1(P) \times \pi_2(P) = \{0,1\} \times \{0,1\} = \{(0,0),(0,1),(1,0),(1,1)\}$$

and hence $P \neq \pi_1(P) \times \pi_2(P)$.

***(10+10 pts)* 6.** Consider the relation $\sim$ on the set $\mathbb{Z}^+$ of positive integers given by

$$a \sim b \text{ if and only if there exists } k \in \mathbb{Z}^+ \text{ such that } ab = k^2$$

for all $a, b \in \mathbb{Z}^+$.
a) Show that $\sim$ is an equivalence relation. We shall prove that $\sim$ is reflexive, symmetric and transitive.

Let $a \in \mathbb{Z}^+$. Then $aa = k^2$ for some $k \in \mathbb{Z}$, namely, $k = a$, and hence we have $a \sim b$. Thus $\sim$ is reflexive.

Let $a, b \in \mathbb{Z}^+$. Assume $a \sim b$. Then, by definition, there exists $k \in \mathbb{Z}^+$ such that $ab = k^2$. Since multiplication of integers is commutative, this means that there exists $k \in \mathbb{Z}^+$ such that $ba = k^2$ and hence, $b \sim a$. Thus $\sim$ is symmetric.

Let $a, b, c \in \mathbb{Z}^+$. Assume $a \sim b$ and $b \sim c$. Then there exist $k, \ell \in \mathbb{Z}^+$ such that $ab = k^2$ and $bc = \ell^2$. It follows that $ac = \left(\frac{k\ell}{b}\right)^2$. Observe that $\frac{k\ell}{b} \in \mathbb{Z}^+$ because the square root of a positive integer is either irrational or a positive integer* and hence, the square root of ac, which is the rational number $\frac{k\ell}{b}$ and which is not irrational, has to be a positive integer. It follows that $a \sim c$. Thus $\sim$ is transitive.

(*: You need not prove this fact here.)

b) Explicitly describe the equivalence class $[2]_\sim$.

Set $S = \{2\ell^2 \mid \ell \in \mathbb{Z}^+\}$. We claim that $[2]_\sim = S$.

For the right-to-left inclusion, let $x \in S$. Then $x = 2\ell^2$ for some $\ell \in \mathbb{Z}^+$. We have $2 \times 2\ell^2 = (2\ell)^2$ and $2\ell \in \mathbb{Z}^+$ and so, by definition, $2 \sim 2\ell$. For the left-to-right inclusion, let $x \in [2]_\sim$. Then $2 \sim x$ and hence there exists $k \in \mathbb{Z}^+$ such that $2x = k^2$. Since $2 \mid k^2$, k has to be even and consequently, $k = 2\ell$ for some $\ell \in \mathbb{Z}^+$. But then, $2x = k^2 = (2\ell)^2$ gives $x = 2\ell^2$. Therefore $x \in S$.

Thus $[2]_\sim = \{2\ell^2 \mid \ell \in \mathbb{Z}^+\} = \{2, 8, 18, 32, \dots\}$