

M E T U Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2025 Midterm I 12 November 2025 17:40		
FULL NAME	STUDENT ID	DURATION 120 MINUTES
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

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(5+5 pts) 1. Let P, Q and R be statements. Show that $(P \wedge Q) \rightarrow R \iff P \rightarrow (Q \rightarrow R)$

a) by using a truth table.

P	Q	R	$\left((P \wedge Q) \rightarrow R \right) \iff \left(P \rightarrow (Q \rightarrow R) \right)$										
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T	T	F	T	F	F
T	F	T	T	F	F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	T	F	T	T	T	F	T	F
F	T	T	F	F	T	T	T	T	F	T	T	T	T
F	T	F	F	F	T	T	F	T	F	T	T	F	F
F	F	T	F	F	F	T	T	T	F	T	F	T	T
F	F	F	F	F	F	T	F	T	F	T	F	T	F

Since $(P \wedge Q) \rightarrow R \iff P \rightarrow (Q \rightarrow R)$ is a tautology, we have $(P \wedge Q) \rightarrow R \iff P \rightarrow (Q \rightarrow R)$.

b) by using standard logical equivalences.

Using the standard logical equivalences, we have that

$$(P \wedge Q) \rightarrow R \iff \neg(P \wedge Q) \vee R \iff (\neg P \vee \neg Q) \vee R \iff \neg P \vee (\neg Q \vee R) \iff \neg P \vee (Q \rightarrow R) \iff P \rightarrow (Q \rightarrow R)$$

Here the first, fourth and fifth equivalences hold because $A \rightarrow B \iff \neg A \vee B$, the second equivalence holds by De Morgan's law that $\neg(A \wedge B) \iff \neg A \vee \neg B$ and the third equivalence holds by the associativity of \vee that $(A \vee B) \vee C \iff A \vee (B \vee C)$.

(10 pts) 2. If exists, write a derivation for the following valid argument.

$$\begin{array}{l} (P \vee Q) \wedge \neg R \\ P \rightarrow R \\ (Q \wedge \neg P) \rightarrow M \\ \hline M \end{array}$$

- 1) $(P \vee Q) \wedge \neg R$
- 2) $P \rightarrow R$
- 3) $(Q \wedge \neg P) \rightarrow M$
- 4) $\neg R$ from 1) and simplification ($A \wedge B \implies B$)
- 5) $P \vee Q$ from 1) and simplification ($A \wedge B \implies A$)
- 6) $\neg R \rightarrow \neg P$ from 2) and contrapositive ($A \rightarrow B \iff \neg B \rightarrow \neg A$)
- 7) $\neg P$ from 4), 6) and Modus Ponens ($(A \rightarrow B) \wedge A \implies B$)
- 8) Q from 5), 7) and $((A \vee B) \wedge \neg A \implies B)$
- 9) $Q \wedge \neg P$ from 7), 8) and adjunction
- 10) M from 3), 9) and Modus Ponens ($(A \rightarrow B) \wedge A \implies B$)

(7×5 pts) 3. In this question, suppose that the possible values of all variables are integers.

a) Suppose that $D(x, y)$ stands for “ x divides y ” and $P(x)$ stands for “ x is prime”. Express the following statement using quantifiers, variables, logical connectives, positive integers, the symbols D and P : “Some integers are divisible by 111 or divisible by every prime integer.”

$$\exists x \left(D(111, x) \vee \forall y (P(y) \rightarrow D(y, x)) \right)$$

b) Find the negation of the following statement and simplify it to a logically equivalent statement so that your final answer does not involve the negation symbol.

$$\exists x \left[\left(\forall y x + y > 0 \right) \rightarrow \left(x = 0 \vee x > 111 \right) \right]$$

By De Morgan’s law, we have the following,

$$\begin{aligned} \neg \exists x \left[\left(\forall y x + y > 0 \right) \rightarrow \left(x = 0 \vee x > 111 \right) \right] &\iff \forall x \left[\neg \left(\left(\forall y x + y > 0 \right) \rightarrow \left(x = 0 \vee x > 111 \right) \right) \right] \\ &\iff \forall x \left[\left(\forall y x + y > 0 \right) \wedge \neg \left(x = 0 \vee x > 111 \right) \right] \\ &\iff \forall x \left[\left(\forall y x + y > 0 \right) \wedge \left(x \neq 0 \wedge x \leq 111 \right) \right] \end{aligned}$$

Therefore, the final simplified statement is $\forall x \left[\left(\forall y x + y > 0 \right) \wedge \left(x \neq 0 \wedge x \leq 111 \right) \right]$.

c) Determine whether or not the following statements are true. In each case, briefly explain your reasoning.

c.i) $\forall x \exists y (x^2 + y < 0)$

This statement is **TRUE**. Because for any integer x , we can choose $y = -x^2 - 1$ in which case we shall have $x^2 + y = x^2 + (-x^2 - 1) = -1 < 0$.

c.ii) $\exists y \forall x (x^2 + y < 0)$

This statement is **FALSE**. We shall show that the negation of this statement, namely,

$$\forall y \exists x (x^2 + y \geq 0)$$

is true. For any integer y , we can choose $x = y$ in which case we shall have

$$x^2 + y = y^2 + y = y(y + 1) \geq 0$$

The last inequality holds because if it were that $y(y + 1) < 0$, then the signs y and $y + 1$ would have to be $+$ and $-$, which is impossible as y and $y + 1$ are consecutive.

c.iii) $\forall x \forall y (x < y \rightarrow (\exists z x < z < y))$

This statement is **FALSE**. For $x = 0$ and $y = 1$, the statement $(0 < 1 \rightarrow (\exists z 0 < z < 1))$ is false as the statement $0 < 1$ is true but the statement $\exists z 0 < z < 1$ is false. (The latter statement is false because there is no integer strictly between 0 and 1.)

In the next question, you are supposed to express various sets in the form $\{x \in A \mid P(x)\}$, where $P(x)$ is written using only logical and mathematical symbols, and in the form a list.

Example. Let A be the set of integers whose squares are less than 17 and which are divisible by 2. This set can be expressed as

$$A = \left\{ n \in \mathbb{Z} \mid n^2 < 17 \wedge 2 \mid n \right\}$$

We can also explicitly list the elements of A as $A = \{-4, -2, 0, 2, 4\}$.

(5+5+5 pts) 4. In Part (a), express the following set in the form $\{x \in A \mid P(x)\}$, where $P(x)$ is written using only logical and mathematical symbols.

a) The set of real numbers which are negative or which are the squares of a rational number

$$\{x \in \mathbb{R} \mid x < 0 \vee \exists q \in \mathbb{Q} x = q^2\}$$

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In Part (b) and (c), express the following sets by explicitly listing their elements.

b) $\left\{ n \in S \mid \forall m \in \mathbb{Z} (m < n \rightarrow m \notin S) \right\} = \{1\}$

Observe that an element n of S satisfies the statement $\forall m \in \mathbb{Z} (m < n \rightarrow m \notin S)$ if each integer less than n is **not** an element of S , which is equivalent by De Morgan's law to saying that there is no integer less than n which is in S . This means that only the minimum element of S can satisfy the formula $P(n) : \forall m \in \mathbb{Z} (m < n \rightarrow m \notin S)$

c) $\left\{ n \in S \mid \exists m \in S (m \neq 1 \wedge m^2 \mid n) \right\} = \{4, 8, 9\}$

The elements of this set are elements of S that are divisible by the square of an element in S which is different than 1. The squares of elements of S that are different than 1 are 4, 9, 16, 25, 36, 49, 64, 81 and the elements of S divisible by some integer in this list are 4, 8, 9.

(10+10+10 pts) 5. In this question, you shall prove several mathematical statement via the indicated proof methods. In each case, make sure that you explain your argument in detail and justify each transition in your proof.

a) Prove the following statement **by a proof by contradiction**: For every real number $x \in [0, \pi/2]$, we have $\sin(x) + \cos(x) \geq 1$.

Assume towards a contradiction that there exists a real number $x \in [0, \pi/2]$ such that $\sin(x) + \cos(x) < 1$. Observe that, since $0 \leq x \leq \pi/2$, we must have $\sin(x) \geq 0$ and $\cos(x) \geq 0$ and hence we obtain the inequality $0 \leq \sin(x) + \cos(x) < 1$. Since we have that $0 \leq a < b$ implies $a^2 < b^2$, squaring the last inequality gives us $(\sin(x) + \cos(x))^2 < 1^2 = 1$. However, as we have $2 \sin(x) \cos(x) \geq 0$, we obtain

$$1 \leq 1 + 2 \sin(x) \cos(x) = \sin^2(x) + 2 \sin(x) \cos(x) + \cos^2(x) = (\sin(x) + \cos(x))^2 < 1$$

which is a contradiction. Therefore, our initial assumption is wrong and consequently, for every real number $x \in [0, \pi/2]$, we have $\sin(x) + \cos(x) \geq 1$.

b) Prove the following statement **by a proof by contrapositive**: Let n be a positive integer. If $n^3 + 2n$ is odd, then n is odd.

We shall prove the contrapositive of the given statement, namely, that “If n is even, then $n^3 + 2n$ is even”.

Assume that n is even. Then $2 \mid n$ and, by definition of divisibility, $n = 2k$ for some integer k . It follows that

$$n^3 + 2n = 8k^3 + 4k = 2(4k^3 + 2k)$$

which implies that $2 \mid n^3 + 2n$ as $4k^3 + 2k$ is an integer. Thus $n^3 + 2n$ is even. This completes the proof of the contrapositive statement.

c) Prove the following statement **by a direct proof**: For every $x \in \mathbb{R}$, if $|x - 3| > 3$, then $x^2 > 6x$.

Let $x \in \mathbb{R}$. Assume that $|x - 3| > 3$. It follows that $x > 6$ or $x < 0$. (Because if it were that $0 \leq x \leq 6$, then we would have $-3 \leq x - 3 \leq 3$ and so, we would have $|x - 3| \leq 3$, which is not the case.)

We now split into two cases:

Case I ($x > 6$). In this case, since $x > 6$, multiplying both sides by x , we obtain $x^2 > 6x$.

Case II. ($x < 0$). In this case, since $x < 0$, we have $6x < 0$ and hence we obtain $x^2 > 0 > 6x$.

In either case, we obtain $x^2 > 6x$.