M E T U Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2025 Final Exam 13 January 2026 13:30		
F U L L N A M E	S T U D E N T I D	DURATION 120 MINUTES
7 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

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In all your answers, you are expected to **explain and justify** your argument, unless specified otherwise.

***(10+7+8 pts)* 1.** Consider the relation $\preccurlyeq$ defined on the set $\mathbb{N} \times \mathbb{N}$ given by

$$(a,b) \preccurlyeq (c,d) \text{ if and only if } a \leq c \text{ and } b \leq d$$

for all $(a,b),(c,d) \in \mathbb{N} \times \mathbb{N}$.

a) Show that $\preccurlyeq$ is a partial order relation.

We shall show that $\preccurlyeq$ is reflexive, antisymmetric and transitive.

- Let $(a,b) \in \mathbb{N} \times \mathbb{N}$. Then, since $a \leq a$ and $b \leq b$, we have $(a,b) \preccurlyeq (a,b)$. Hence $\preccurlyeq$ is reflexive.
- Let $(a,b),(c,d) \in \mathbb{N} \times \mathbb{N}$. Assume that $(a,b) \preccurlyeq (c,d)$ and $(c,d) \preccurlyeq (a,b)$. Then, by definition, we have $a \leq c$ and $b \leq d$, and, $c \leq a$ and $d \leq b$. It follows from $a \leq c$ and $c \leq a$ that $a = c$. Similarly, it follows from $b \leq d$ and $d \leq b$ that $b = d$. Thus $(a,b) = (c,d)$. This shows that $\preccurlyeq$ is antisymmetric.
- Let $(a,b),(c,d),(e,f) \in \mathbb{N} \times \mathbb{N}$. Assume that $(a,b) \preccurlyeq (c,d)$ and $(c,d) \preccurlyeq (e,f)$. Then, by definition, we have $a \leq c$ and $b \leq d$, and, $c \leq e$ and $d \leq f$. It follows from $a \leq c$ and $c \leq e$ that $a \leq e$. Similarly, it follows from $b \leq d$ and $d \leq f$ that $b \leq f$. Since $a \leq e$ and $b \leq f$, by definition, we have $(a,b) \preccurlyeq (e,f)$. This shows that $\preccurlyeq$ is transitive.

Therefore, $\preccurlyeq$ is a partial order relation.

b) Determine whether or not $\preccurlyeq$ is a total (that is, linear) order relation.

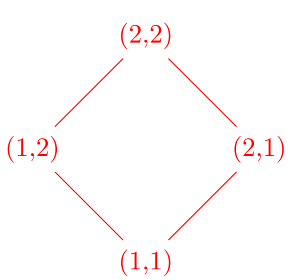
Observe that $(1,4) \not\preccurlyeq (2,3)$ since $4 \not\leq 3$, and that $(2,3) \not\preccurlyeq (1,4)$ since $2 \not\leq 1$. Therefore, not any two elements in $\mathbb{N} \times \mathbb{N}$ are comparable with respect to $\preccurlyeq$. This means that $\preccurlyeq$ is not a total order relation.

Set $X = \{1,2\} \times \{1,2\}$. Consider the restriction $\preccurlyeq_X$ of the partial order relation $\preccurlyeq$ to the subset $X \subseteq \mathbb{N} \times \mathbb{N}$, that is, $\preccurlyeq_X$ is the relation on X given by

$$(a,b) \preccurlyeq_X (c,d) \text{ if and only if } a \leq c \text{ and } b \leq d$$

for all $(a,b),(c,d) \in X$.

c) Draw the Hasse diagram of the partially ordered set $(X, \preccurlyeq_X)$.



(8+8+8 pts) **2.** Given a map $f : \mathbb{Z} \to \mathbb{Z}$ and $k \in \mathbb{Z}^+$, the map f is said to be *k-periodic* in the case that $f(n) = f(n+k)$ for all $n \in \mathbb{Z}$. For example, the map $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = (-1)^n$ is 2-periodic. For each $k \in \mathbb{Z}^+$, let Per_k denote the set of all k-periodic maps on $\mathbb{Z}$, that is,

$$\mathrm{Per}_k = \{f : \mathbb{Z} \to \mathbb{Z} \mid f \text{ is } k\text{-periodic}\}$$

a) Let $k \in \mathbb{Z}^+$ and set $\mathbb{Z}^k = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{k \text{ times}}$. Consider the function $\Psi : \mathrm{Per}_k \to \mathbb{Z}^k$ given by

$$\Psi(f) = (f(0), \ldots, f(k-1))$$

for all $f \in \mathrm{Per}_k$. Show that Ψ is a bijection. Let us first show that Ψ is injective. Let $f, \widehat{f} \in \mathrm{Per}_k$. Assume that $\Psi(f) = \Psi(\widehat{f})$. Then we have $(f(0), \ldots, f(k)) = (\widehat{f}(0), \ldots, \widehat{f}(k))$. Let $n \in \mathbb{Z}$. Then $n = mk + r$ for some $m, r \in \mathbb{Z}$ with $0 \le r < k$, in other words, m is the quotient and r is the remainder of diving n by k. But then, by the periodicity of f and $\widehat{f}$, as the values of f and $\widehat{f}$ do not change whenever we add or subtract k to their inputs, we have

$$\widehat{f}(n) = \widehat{f}(r+mk) = \widehat{f}(r+(m-1)k) = \cdots = \widehat{f}(r+k) = \widehat{f}(r) = f(r) = f(r+k) = \cdots = f(r+(m-1)k) = f(r+mk) = f(n)$$

Note that the equality $\widehat{f}(r) = f(r)$ holds because $r \in \{0, \ldots, k-1\}$ and $(f(0), \ldots, f(k)) = (\widehat{f}(0), \ldots, \widehat{f}(k))$. Since $f(n) = \widehat{f}(n)$ for all $n \in \mathbb{Z}$, we have $f = \widehat{f}$ and hence, Ψ is injective.
Let us now show that Ψ is surjective. Let $\mathbf{a} = (a_0, \ldots, a_{k-1}) \in \mathbb{Z}^k$. Consider the function $f_{\mathbf{a}} : \mathbb{Z} \to \mathbb{Z}$ given by

$$f_{\mathbf{a}}(n) = a_r \text{ where } n \text{ can be written as } n = mk + r \text{ for some } m, r \in \mathbb{Z} \text{ with } 0 \le r < k$$

(Note that the division algorithm produces unique $m, r \in \mathbb{Z}$ with $0 \le r < k$, therefore the function $f_{\mathbf{a}}$ is well-defined. Though, you need not argue this fact here.) Observe that, given $n \in \mathbb{Z}$, we can write n as $n = mk + r$ for some $m, r \in \mathbb{Z}$ with $0 \le r < k$ and consequently,

$$f_{\mathbf{a}}(n) = f_{\mathbf{a}}(mk + r) = a_r = f_{\mathbf{a}}((m+1)k + r) = f_{\mathbf{a}}(mk + r + k) = f_{\mathbf{a}}(n + k)$$

Thus the map $f_{\mathbf{a}}$ is k-periodic and so $f_{\mathbf{a}} \in \mathrm{Per}_k$. But now, we have

$$\Psi(f_{\mathbf{a}}) = (f_{\mathbf{a}}(0), \ldots, f_{\mathbf{a}}(k-1)) = (a_0, \ldots, a_{k-1}) = \mathbf{a}$$

This shows that Ψ is surjective.

A map $f : \mathbb{Z} \to \mathbb{Z}$ is said to be *periodic* in the case that it is k-periodic for some $k \in \mathbb{Z}^+$. Let Per denote the set of periodic maps on $\mathbb{Z}$, that is, $\mathrm{Per} = \{f : \mathbb{Z} \to \mathbb{Z} \mid f \text{ is periodic}\}$.
b) Show that Per_k is countable for all $k \in \mathbb{Z}^+$ and that Per is countable.
Recall that $\mathbb{Z}$ is countable and a finite product of countable sets is countable. Hence $\mathbb{Z}^k$ is countable for all $k \in \mathbb{Z}^+$. By Part (a), $\mathrm{Per}_k \sim \mathbb{Z}^k$ and consequently, Per_k is countable for all $k \in \mathbb{Z}^+$. Finally, since a countable union of countable sets is countable and $\mathrm{Per} = \bigcup_{k \in \mathbb{Z}^+} \mathrm{Per}_k$, we have that Per is countable.
Let Func denote the set of all maps on $\mathbb{Z}$, that is, $\mathrm{Func} = \{f : \mathbb{Z} \to \mathbb{Z} \mid f \text{ is a function}\}$. You are given that $\mathrm{Func} \sim \mathcal{P}(\mathbb{Z})$.
c) Determine whether or not the set of non-periodic functions on $\mathbb{Z}$ is countable.
Consider the set of non-periodic functions $\mathrm{NPer} = \{f : \mathbb{Z} \to \mathbb{Z} \mid f \text{ is non-periodic}\}$. We shall show that NPer is uncountable. Assume towards a contradiction that NPer is countable. Then, since Per is countable by Part (b) and a union of two countable sets is countable, the set $\mathrm{Func} = \mathrm{Per} \cup \mathrm{NPer}$ is countable. Consequently, from the given fact $\mathrm{Func} \sim \mathcal{P}(\mathbb{Z})$, we obtain that $\mathcal{P}(\mathbb{Z})$ is countable. This contradicts Cantor's theorem that $\mathbb{Z} \prec \mathcal{P}(\mathbb{Z})$.

***(8 pts)* 3.** Using induction, prove that $2^n + n^2 < 3^n$ for all integers $n \geq 4$.

(Note: There are essentially infinite many different ways to do this induction. The method below is simply *one* way to do this.)
Let $P(n)$ denote the statement $2^n + n^2 < 3^n$. We shall show that $P(n)$ holds for all integers $n \geq 4$ by the principle of mathematical induction.
Base step: For $n = 4$, we have that $2^4 + 4^2 = 32 < 81 = 3^4$. Thus $P(n)$ holds for $n = 4$.
Inductive step: Let $n \geq 4$ be an integer. Suppose that $P(n)$. We wish to show that $P(n+1)$ holds under this assumption. By the inductive assumption $P(n)$, we have $2^n < 3^n - n^2$ and consequently,

$$2^{n+1}+(n+1)^2 = 2\cdot 2^n+n^2+2n+1 < 2\cdot(3^n-n^2)+n^2+2n+1 = 2\cdot 3^n-(n^2-2n-1) < 2\cdot 3^n < 3\cdot 3^n = 3^{n+1}$$

Observe that the inequality second from last holds because

$$n^2 - 2n - 1 > n^2 - 2n - 2 = n^2 - 2(n-1) > n^2 - (n-1)(n-1) = 2n - 1 > 0$$

and hence $-(n^2 - 2n - 1) < 0$. We have obtained that $P(n+1)$ holds. Therefore, by the principle of mathematical induction, $P(n)$ holds for all integers $n \geq 4$.

***(5+5 pts)* 4.** Consider the relation $\trianglelefteq$ on the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ given by

$$i \trianglelefteq j \text{ if and only if } \frac{i}{j} = 2^k \text{ for some } k \in \mathbb{N}, \text{ where } \mathbb{N} = \{0, 1, 2, 3 \dots\}.$$

for all $i, j \in \mathbb{Z}^+$. You are given that $\trianglelefteq$ *is* a partial order relation.
a) Show that $\trianglelefteq$ *is not* an equivalence relation.

Observe that $2 \trianglelefteq 1$ since $\frac{2}{1} = 2^1$, however, $1 \ntrianglelefteq 2$ since $\frac{1}{2} \neq 2^k$ for any $k \in \mathbb{N}$. Since we have $2 \trianglelefteq 1$ and $1 \ntrianglelefteq 2$, the relation $\trianglelefteq$ is not symmetric. Therefore, $\trianglelefteq$ is not an equivalence relation.

In the rest of the question, consider the partially ordered set $(\mathbb{Z}^+, \trianglelefteq)$.
b) Find a non-empty finite subset $A \subseteq \mathbb{Z}^+$ such that A has no least upper bound and no greatest lower bound.

Choose $A = \{1, 3\}$. Assume towards a contradiction that A has a least upper bound, say, $p \in \mathbb{Z}^+$. Then p is an upper bound for A. Consequently, $1 \trianglelefteq p$ and $3 \trianglelefteq p$. These imply that $\frac{p}{1} = 2^k$ for some $k \in \mathbb{N}$ and $\frac{p}{3} = 2^\ell$ for some $\ell \in \mathbb{N}$. Combining these, we obtain that $3 = 2^{k-\ell}$, which is a contradiction. Similarly, assume towards a contradiction that A has a greatest lower bound, say, $q \in \mathbb{Z}^+$. Then q is a lower upper bound for A. Consequently, $q \trianglelefteq 1$ and $q \trianglelefteq 3$. These imply that $\frac{1}{q} = 2^m$ for some $m \in \mathbb{N}$ and $\frac{3}{q} = 2^n$ for some $n \in \mathbb{N}$. Combining these, we obtain that $3 = 2^{n-m}$, which is a contradiction.

***(8 pts)* 5.** Let X be a nonempty set and R be a relation on X. Let Δ_X denote the relation

$$\Delta_X = \{(x, y) \in X \times X \mid x = y\}$$

Show that if the relation R is symmetric, antisymmetric and reflexive simultaneously, then $R = \Delta_X$.
Assume that the relation R is symmetric, antisymmetric and reflexive . We shall show that $R \subseteq \Delta_X$ and $\Delta_X \subseteq R$. Let $(x, y) \in R$. Since $(x, y) \in R$ and R is symmetric, we have $(y, x) \in R$. But now, since R is antisymmetric and $(x, y) \in R$ and $(y, x) \in R$, we have $x = y$ and hence, by definition, $(x, y) \in \Delta_X$. This shows $R \subseteq \Delta_X$. Now let $(x, y) \in \Delta_X$. Then, by definition, $x = y$. Since R is reflexive and $x \in X$, $(x, y) = (x, x) \in R$. This shows $\Delta_X \subseteq R$.

(5+5+5+5=20 pts) **6.** You are given some proofs of theorems below. These given proofs **may be correct or incorrect**. If a proof is correct, then write **"the proof is correct"** and indicate the name of the proof method used. If a proof is incorrect, then write **"the proof is incorrect"** and **briefly explain the mistake** in the proof.

Theorem. There exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$ we have $x^2 + y \geq 0$.

Proof. Let $x \in \mathbb{Z}$. Choose $y = |x|$. Observe that $x^2 \geq 0$ by the properties of arithmetic operations and $|x| \geq 0$ by the properties of the absolute value. Consequently, adding these inequalities side-by-side, we obtain $x^2 + y = x^2 + |x| \geq 0$.

Answer: The proof is incorrect. The mistake is a standard mistake in understanding the order of quantifiers. The attempted proof shows that for each $x \in \mathbb{Z}$ there exists $y \in \mathbb{Z}$, depending on x, such that $x^2 + y \geq 0$ holds. In other words, the attempted proof shows that $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ x^2 + y \geq 0$. However, in a correct proof, we should construct a single value $y \in \mathbb{Z}$ that simultaneously works for all $x \in \mathbb{Z}$ so that we will have proven that $\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z}\ x^2 + y \geq 0$.

Theorem. The relation $\sim$ on the set $\mathbb{Z}^+$ of positive integers given by

$$a \sim b \text{ if and only if there exists } k \in \mathbb{Z}^+ \text{ such that } ab = k^2$$

is transitive.

Proof. Let $a, b, c \in \mathbb{Z}^+$. Suppose that $a \sim b$ and $b \sim c$. Then, by the definition of $\sim$, there exists $k \in \mathbb{Z}^+$ such that $ab = k^2$ and $bc = k^2$. It follows that $ab = k^2 = bc$ and hence, $a = c$. But then, we have that $ac = a^2$ and so, by the definition of $\sim$, we have $a \sim c$. This shows that $\sim$ is transitive.

Answer: The proof is incorrect. The mistake is a standard mistake in understanding the role of dummy variables. From the assumptions $a \sim b$ and $b \sim c$, we obtain that there exist $k, \ell \in \mathbb{Z}^+$ such that $ab = k^2$ and $bc = \ell^2$. However, it need not be the case that $k = \ell$. The variable k in the definition of $\sim$ is simply a dummy variable and, for each instance of the relation $\sim$ holding, we get possibly a different integer depending on the integers that are related under $\sim$.

Theorem. The function $\sigma : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ given by $\sigma(A) = \{a - 1 \mid a \in A\}$ is injective.

Proof. Let $a, b \in \mathbb{Z}$. Assume that $\sigma(a) = \sigma(b)$. Then, by definition, we have $a - 1 = \sigma(a) = \sigma(b) = b - 1$. Since $a - 1 = b - 1$, by adding 1 to both sides, we obtain $a = b$. We have shown that $\sigma(a) = \sigma(b)$ implies $a = b$ for any $a, b \in \mathbb{Z}$. Thus σ is injective.

Answer: The proof is incorrect. The mistake in the attempted proof is that the function σ is applied to elements of $\mathbb{Z}$, which is not the domain of σ, instead of elements of $\mathcal{P}(\mathbb{Z})$, which is the domain of σ.

Theorem. Let A, B, C, D be sets. Then $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof. Let $(x, y) \in (A \times B) \cup (C \times D)$. Then $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Since $A \subseteq A \cup C$ and $B \subseteq B \cup D$, we have $A \times B \subseteq (A \cup C) \times (B \cup D)$. This subset inclusion now gives $(x, y) \in (A \cup C) \times (B \cup D)$ as we already have $(x, y) \in A \times B$. This completes the proof that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Answer: The proof is incorrect. The mistake is a standard mistake in understanding set operations. From the assumption $(x, y) \in (A \times B) \cup (C \times D)$, one can only deduce that $(x, y) \in A \times B$ or $(x, y) \in C \times D$. However, the attempted proof connects these two statement by "and" instead of "or".

(5 pts) **7.** The following proof is a **correct** proof of a theorem whose statement is left incomplete. Complete the statement of the theorem appropriately.

Theorem. Let X, Y, Z be sets. If $Y \subseteq X$ or $Y \subseteq Z$, then $Y \in \mathcal{P}(X) \cup \mathcal{P}(Z)$.

Proof. We shall prove this statement by contrapositive. Suppose that $Y \notin \mathcal{P}(X) \cup \mathcal{P}(Z)$. Note that, by the definition of union, we have $Y \in \mathcal{P}(X) \cup \mathcal{P}(Z)$ iff $Y \in \mathcal{P}(X)$ or $Y \in \mathcal{P}(Z)$. Since $Y \notin \mathcal{P}(X) \cup \mathcal{P}(Z)$, by negating the right hand side of this if-and-only-if statement via De Morgan's law, we obtain $Y \notin \mathcal{P}(X)$ and $Y \notin \mathcal{P}(Z)$. Then, by the definition of power set, we obtain $Y \not\subseteq X$ and $Y \not\subseteq Z$. ■