27 February 2020

Thursday, February 27, 2020 8:41 AM

Example: let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Find $f'(x)$.

Solution: For $x \neq 0$, we have $f'(x) = \left(x^2 \sin\left(\frac{1}{x}\right)\right)' = 2x \sin\left(\frac{1}{x}\right) + \cancel{x^2} \cdot \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{\cancel{x^2}}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

For $x=0$, we have $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\cancel{2}} \sin\left(\frac{1}{h}\right) - 0}{\cancel{h}} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

by the squeeze theorem
$0 \leq \left|\sin\left(\frac{1}{h}\right)\right| \leq 1$
$0 \leq \left|h \sin\left(\frac{1}{h}\right)\right| \leq |h|$
and $\lim_{h\to 0} |h| = 0$, so
$\lim_{h\to 0} \left|h \sin\left(\frac{1}{h}\right)\right| = 0$ so
$\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$

Thus

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

WARNING: Note that $\lim_{x \to 0} f'(x) = \lim_{x\to 0} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ d.n.e

$\not\parallel$

but $f'(0) = 0$

Example: let $g(x) = \sqrt[3]{x} + \sec(x^2+1)$. Find an equation of the tangent line to $y = g(x)$ at $x=0$.

Solution: We have $g'(x) = \frac{1}{3} \frac{1}{\sqrt[3]{x^2}} + \sec(x^2+1)\tan(x^2+1) \cdot 2x$ for $x \neq 0$. Indeed, one can show that $g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h} + \sec(h^2+1) - \sec(0^2+1)}{h} = +\infty$ ← can be computed (not easily seen!)

This means that $y = g(x)$ has a vertical tangent line at $x=0$. So it is $x = 0$

To recap: If $g'(x)$ exists (that is, is a real number), it gives us the slope of the $\boxed{\text{non-vertical}}$ tangent line to the graph of g at x.

If $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \mp\infty$, then this means that the graph of g has a $\boxed{\text{vertical}}$ tangent line at x.

Higher-order derivatives

Given a function f, by iteratively applying the differentiation operation, one can define the derivative of f of order n, for each natural number n.

$$f(x) = f^{(0)}(x)$$
$$f'(x) = f^{(1)}(x) = \frac{df}{dx}$$
$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$$
$$f'''(x) = f^{(3)}(x) = \frac{d^3 f}{dx^3}$$
$$\vdots$$

$f^{\overbrace{'''\cdots'}^{n \text{ times}}}(x) = f^{(n)}(x)$ the n-th order derivative of f

Example:

- $$f(x) = \sin(x) = f^{(4)}(x)$$
  $$f'(x) = \cos(x) = f^{(5)}(x)$$
  $$f''(x) = -\sin(x) \quad \vdots$$
  $$f'''(x) = -\cos x$$

- $$f(x) = (1+x)^{-1}$$
  $$f'(x) = -1\,(1+x)^{-2}$$
  $$f''(x) = (-1)(-2)(1+x)^{-3}$$
  $$f'''(x) = (-1)(-2)(-3)(1+x)^{-4}$$
  $$\vdots$$
  can be proven by induction!
  $$f^{(n)}(x) = (-1)^n n!\,(1+x)^{-(n+1)}$$

- Let $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. We've shown that

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and so}$$

$$(f')'(x) = f''(x) = \begin{cases} \left(2\sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}\right) - \left(-\sin\left(\frac{1}{x}\right)\right)\cdot\frac{-1}{x^2} & \text{if } x \neq 0 \\ \lim_{h\to 0}\frac{f'(h)-f'(0)}{h} = \lim_{h\to 0}\frac{2h\sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right) - 0}{h} \quad \text{d.n.e} & \text{if } x = 0 \end{cases}$$

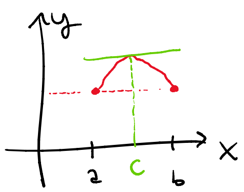
Remark: It is possible to have functions f such that $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ all exist but $f^{(n+1)}$ does not exist at a point x.

Our next objective is to prove the mean value theorem. We shall need the following.

Theorem: Let f be defined on (a,b) and suppose that $f'(c)$ exists and f has a maximum (or minimum) at c. Then $f'(c) = 0$.

Proof: Since f has a max at c, we have $f(x) \leq f(c)$ for all x in (a,b). So, for $c < x < b$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$ and, for $a < x < c$, we have $\frac{f(x)-f(c)}{x-c} \geq 0$. As $\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$ exists, we must have $f'(c) \leq 0$ and $f'(c) \geq 0$. Thus $f'(c) = 0$. (The proof for the minimum case is similar.)

Rolle's theorem: Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = f(b)$. Then there is c in (a,b) such that $f'(c) = 0$.



Proof: Case I: (For all x in $[a,b]$, $f(x) = f(a) = f(b)$.)
Then choose c as any point in $[a,b]$ and $f'(c) = 0$.

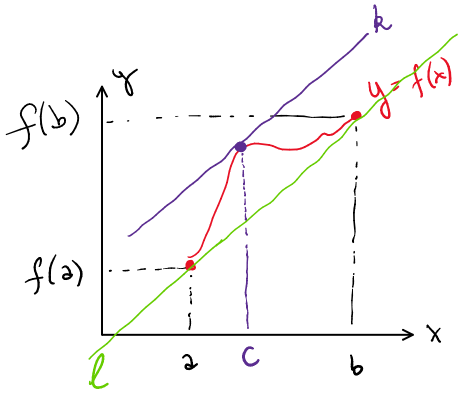
Case II: (There is some x in $[a,b]$, $f(x) \neq f(a) = f(b)$.)

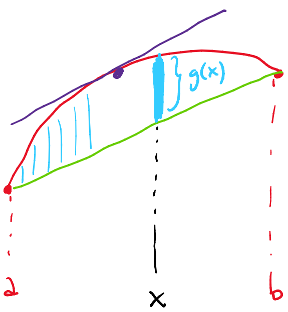
For now, suppose that $f(x) > f(a)$. By the max-min theorem, there is c in $[a,b]$ such that f has a maximum at c. Note that $c \neq a$ and $c \neq b$. But then, as f is differentiable on (a,b), by the prev. theorem, we must have $f'(c) = 0$.

(The proof for the case that $f(x) < f(a)$ is similar.)

Mean Value Theorem: Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there is some c in (a,b) such that

the slope of k $\quad f'(c) = \dfrac{f(b) - f(a)}{b-a}\quad$ the slope of ℓ





Proof: Set
$$g(x) = f(x) - \left(f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\right).$$

Then g is cont. on $[a,b]$ and differentiable on (a,b). Observe that $\boxed{g(b) = 0 = g(a)}$. By Rolle's theorem, there is c in (a,b) such that $g'(c) = 0$. So
$$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$
and thus
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Theorem: Let f be differentiable on (a,b). Then

- If $f'(x) > 0$ for all x in (a,b), then f is increasing on (a,b).
- — $f'(x) < 0$ — decreasing —
- — $f'(x) \geq 0$ — non-decreasing —
- — $f'(x) \leq 0$ — non-increasing —

Proof: Use MVT (or read your book!)

Example: Show that $\sin x < x$ for all $x > 0$.

Solution: Note that $\sin$ is differentiable everywhere. For $0 < x \leq 1$, by applying MVT to $\sin$ on the interval $[0, x]$, we see that there exist $0 < c < x$ such that

$$\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x - 0} = \frac{d}{dx} \sin x \Big|_{x=c} = \cos c < 1 \quad \text{and so} \quad \sin x < x.$$

For $x > 1$, we clearly have $\sin x < x$. Thus $\sin x < x$ for all $x > 0$.