25 February 2020

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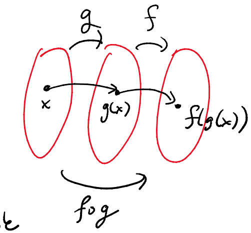
Recall that if f and g are differentiable at c, then so are $f \mp g$, $f \cdot g$ and $\frac{f}{g}$ (provided that $g(c) \neq 0$).

What about $f \circ g$ or $g \circ f$?



Theorem (The Chain Rule): Suppose g is differentiable at x and f is differentiable at $g(x)$. Then $f \circ g$ is differentiable at x and moreover,

$$(f \circ g)'(x) = \left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

Example: $\frac{d}{dx}\left(\left((x^2+1)^3 - x^2\right)^2\right)$

In full detail: $g(x) = x^2+1$, $h(x) = x^2$, $f(x) = x^3$

$(x^2+1)^3 = f(g(x))$

$(x^2+1)^3 - x^2 = f(g(x)) - h(x)$

$\left((x^2+1)^3 - x^2\right)^2 = h\left(f(g(x)) - h(x)\right)$

$$\left(h(f(g(x)) - h(x))\right)' = h'\left(f(g(x)) - h(x)\right) \cdot \left(f(g(x)) - h(x)\right)'$$
$$= 2\left((x^2+1)^3 - x^2\right) \cdot \left([f(g(x))]' - h'(x)\right)$$
$$= 2\left((x^2+1)^3 - x^2\right) \cdot \left(f'(g(x)) \cdot g'(x) - 2x\right)$$
$$= 2\left((x^2+1)^3 - x^2\right) \cdot \left(3(x^2+1)^2 \cdot 2x - 2x\right)$$

Example: $\frac{d}{dx}\left(\sin^2(\cos(x^2+1) \cdot x)\right) = 2\left(\sin(\cos(x^2+1) \cdot x)\right) \cdot \left(\sin(\cos(x^2+1) \cdot x)\right)'$

[For now, suppose that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.]

$$= 2\sin(\cos(x^2+1) \cdot x) \cdot \cos(\cos(x^2+1) \cdot x) \cdot (\cos(x^2+1) \cdot x)'$$
$$= 2\sin(\cos(x^2+1) \cdot x) \cdot \cos(\cos(x^2+1) \cdot x) \cdot \left((\cos(x^2+1))' \cdot x + \cos(x^2+1) \cdot 1\right)$$
$$= 2\sin(\cos(x^2+1) \cdot x) \cdot \cos(\cos(x^2+1) \cdot x) \cdot \left(-\sin(x^2+1) \cdot 2x \cdot x + \cos(x^2+1)\right)$$

Example: Let f, g be differentiable functions such that $f'(2) = 5$, $g(2) = 1$, $f(3) = 7$, $g'(2) = 1$, $f'(1) = 3$, $f'(4) = 8$, $f(1) = 1$. Set $h(x) = f(f(g(x)) + g(x))$. Find $h'(2)$.

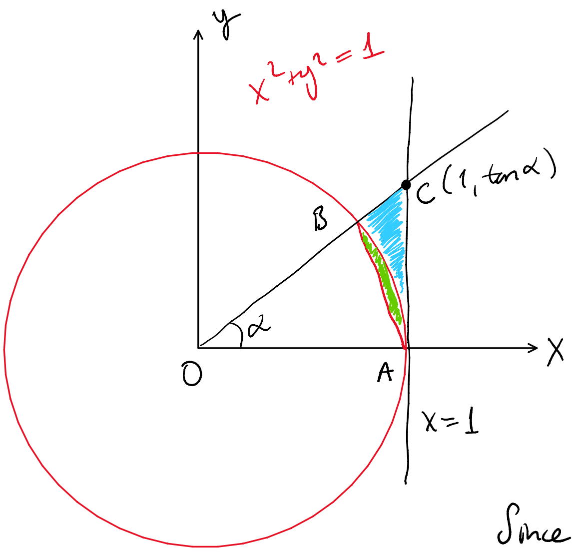
Solution: By the chain rule, we have

$$h'(x) = f'(f(g(x)) + g(x)) \cdot (f(g(x)) + g(x))'$$
$$= f'(f(g(x)) + g(x)) \cdot \left(f'(g(x)) \cdot g'(x) + g'(x)\right)$$

Plugging in $x=2$ gives

$$h'(2) = f'(f(g(2))+g(2)) \cdot (f'(g(2)) \cdot g'(2) + g'(2))$$
$$= f'(f(1)+1) \cdot (f'(1) \cdot 1 + 1)$$
$$= f'(1+1) \cdot (3 \cdot 1 + 1) = 5 \cdot (3+1) = 20$$

Derivatives of trigonometric functions



For $0<\alpha<\frac{\pi}{2}$, we have that

$$\text{Area}(\triangle AOB) < \text{Area sector } AOB < \text{Area}(\triangle AOC)$$

$$\cancel{\frac{1}{2}\cdot 1 \cdot 1 \cdot} \sin(\alpha) < \cancel{\pi} \cdot \frac{\alpha}{2\cancel{\pi}} < \frac{\tan\alpha}{2} = \frac{\sin\alpha}{\cancel{2}\cos\alpha}$$

Dividing all sides by $\frac{\sin\alpha}{2} > 0$, we get

$$1 < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha} \quad \text{for } 0<\alpha<\frac{\pi}{2}$$

Since $\lim\limits_{\alpha\to 0^+} \frac{1}{\cos\alpha} = 1$, by the squeeze theorem, $\lim\limits_{\alpha\to 0^+} \frac{\alpha}{\sin\alpha} = 1$ and hence $\lim\limits_{\alpha\to 0^+} \frac{\sin\alpha}{\alpha} = 1$. By a similar argument, one can show that $\lim\limits_{\alpha\to 0^-} \frac{\sin\alpha}{\alpha} = 1$. Therefore

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Example: Show that $\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0$

Solution:

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{(\cancel{1}-2\sin^2(\frac{x}{2})) \cancel{-1}}{x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2(\frac{x}{2})}{x} = \lim_{x\to 0} -1 \cdot \frac{\sin(\frac{x}{2})}{\frac{x}{2}} \cdot \sin\left(\frac{x}{2}\right)$$

$$= -1 \cdot 1 \cdot 0 = 0$$

$$\boxed{\begin{aligned} \sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \end{aligned}}$$

Set $t = \frac{x}{2}$. Then

$$\lim_{x\to 0} \frac{\sin(\frac{x}{2})}{\frac{x}{2}} = \lim_{t\to 0} \frac{\sin(t)}{t} = 1$$

Thus

- $$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h\to 0} \sin x \left(\overset{\nearrow 0}{\cancel{\frac{\cos h - 1}{h}}}\right) + \cos x \cdot \overset{\nearrow 1}{\cancel{\frac{\sin h}{h}}}$$
$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

$\sin x \cos \frac{\pi}{2} + \cos x \sin \frac{\pi}{2}$

- $$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(\sin\left(x+\frac{\pi}{2}\right)\right) = \cos\left(x+\frac{\pi}{2}\right) \cdot \left(x+\frac{\pi}{2}\right)' = \cos\left(x+\frac{\pi}{2}\right) \cdot 1$$
$$= \cancel{\cos x \cos \frac{\pi}{2}} - \sin x \sin \frac{\pi}{2}$$
$$= -\sin x$$

- $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$
$$= 1 + \tan^2 x$$

- $$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$
$$= -(1+\cot^2 x)$$

- $$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

- do this!
$$\frac{d}{dx}(\operatorname{cosec} x) = \ldots\ldots\ldots = -\operatorname{cosec} x \cdot \cot x$$

Example: Find an equation of the tangent line to the curve given by $y = \tan(\cos(x)) + x$ at the point $(\frac{\pi}{2}, 0)$.

Solution: The slope of this line is $y'(\frac{\pi}{2})$. So it is

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \left.\left(\sec^2(\cos(x)) \cdot (-\sin x) + 1\right)\right|_{x=\frac{\pi}{2}} = (1 \cdot (-1)) + 1 = 0$$

Thus its equation is $y - 0 = 0 \cdot (x - \frac{\pi}{2}) = 0$, that is, $y = 0$.

Example: Find $\left.\dfrac{d\,f(\sin x)}{dx}\right|_{x=0}$ where $f(x)=\begin{cases} \dfrac{\tan^2 x}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

Solution: By the chain rule,

$$\frac{d}{dx} f(\sin x) = f'(\sin x)\,.\,\cos x$$

So $\left.\dfrac{d}{dx} f(\sin x)\right|_{x=0} = f'(\sin 0)\,.\,\cos 0 = f'(0)$. We now compute $f'(0)$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{\tan^2 h}{h} - 0}{h} = \lim_{h\to 0} \frac{\tan^2 h}{h^2}$$

$$= \lim_{h\to 0} \frac{\sin h}{\cos h}\cdot\frac{1}{h}\cdot\frac{\sin h}{\cos h}\cdot\frac{1}{h} = 1$$

So $\left.\dfrac{d}{dx} f(\sin(x))\right|_{x=0} = 1$