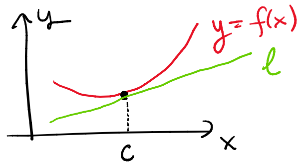
21 February 2020

Thursday, February 20, 2020 8:45 AM

Recall that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{t\to x} \frac{f(t) - f(x)}{t-x}$$

Geometrically speaking, if f' exists at a point c, then there is a $\boxed{\text{tangent line}}$ to $y=f(x)$ at $x=c$ and its slope is $f'(c)$



$m_\ell = f'(c)$

Notations for derivatives: $f'(x) = \dfrac{d\,f(x)}{dx} \left(= \dot{f}(x) = D_x f(x)\right)$

Example: Find $\dfrac{d\,(x^n)}{dx}$ where n is a positive integer.

$$\binom{n}{i} = \frac{n(n-1)(n-2)\cdots(n-(i-1))}{i\,(i-1)(i-2)\cdots 1}$$

Solution:

$$\frac{d\,(\overbrace{x^n}^{f(x)})}{dx} = \lim_{h\to 0} \frac{\overbrace{(x+h)^n}^{f(x+h)} - \overbrace{x^n}^{f(x)}}{h} = \lim_{h\to 0} \frac{\left(\cancel{x^n} + \binom{n}{1}x^{n-1}\cancel{h} + \binom{n}{2}x^{n-2}h^{\cancel{2}} + \cdots + \binom{n}{n-1}x^1 h^{\cancel{n-1}} + h^{\cancel{n}}\right) - \cancel{x^n}}{\cancel{h}}$$

$$= \lim_{h\to 0} \binom{n}{1}x^{n-1} + \cancel{\binom{n}{2}x^{n-2}h} + \cancel{\cdots} + \cancel{\binom{n}{n-1}x^1 h^{n-2}}$$

$$= n x^{n-1}$$

Example: let $f(x) = \begin{cases} x^2 \cdot \cos\left(\frac{1}{x}\right) + 1 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. Find $f'(0)$ if exists.

Wrong solution: $f'(x) = \begin{cases} 2x\cos\left(\frac{1}{x}\right) + x^2 \cdot -\sin\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} + 0 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. So $f'(0) = 0$.

Wrong solution II: $f'(x) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$ if $x \neq 0$. So $f'(0) = \underbrace{\lim_{x\to 0}}_{\text{d.n.e!}} f'(x)$.

Solution:

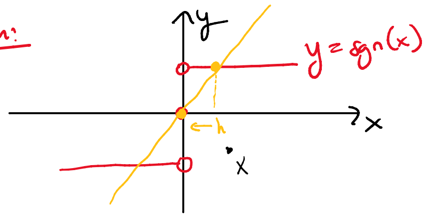
$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}} \cdot \cos\left(\frac{1}{h}\right) + \cancel{1} - \cancel{1}}{\cancel{h}} = \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$$

apply the squeeze theorem

Example: Let $sgn(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$. Find $sgn'(x)$.

Solution:



Let $x \in \mathbb{R}$.

Case I: $(x>0)$

$$sgn'(x) = \lim_{h\to 0} \frac{sgn(x+h)-sgn(x)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \underline{\underline{0}}$$

Case II: $(x<0)$ $\quad sgn'(x) = \lim_{h\to 0} \frac{sgn(x+h)-sgn(x)}{h} = \lim_{h\to 0}\frac{-1-(-1)}{h} = \underline{\underline{0}}$

Case III: $(x=0)$

$$sgn'(0) = \lim_{h\to 0} \frac{sgn(0+h)-sgn(0)}{h} = \lim_{h\to 0} \frac{sgn(h)}{h}$$

$$\lim_{h\to 0^+} \frac{sgn(h)}{h} = \lim_{h\to 0^+} \frac{1}{h} = \underline{\underline{+\infty}}$$

$$\lim_{h\to 0^-} \frac{sgn(h)}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \underline{\underline{+\infty}}$$

$$sgn'(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \text{d.n.e.} & \text{if } x=0 \end{cases}$$

Differentiation rules

Theorem: If f is differentiable at $x=c$, then f is continuous at $x=c$.

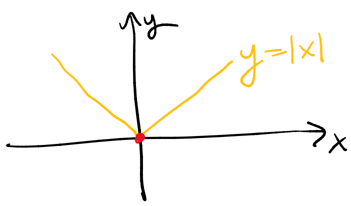
Proof: Suppose that f is differentiable at $x=c$. Then $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ exists and so,

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \cdot \underbrace{\lim_{x\to c} x-c}_{0} = \lim_{x\to c} \frac{f(x)-f(c)}{\cancel{x-c}} \cdot \cancel{x-c}$$

$$= \lim_{x\to c} f(x)-f(c) = 0$$

Thus $\lim_{x\to c} f(x) = f(c)$ and so f is continuous at c.

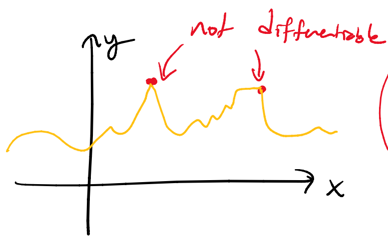
WARNING: continuity $\not\Rightarrow$ differentiability



$|x|$ is not differentiable at $x=0$ as

$$\left.\frac{d\,|x|}{dx}\right|_{x=0} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} \text{ d.n.e}$$

$$\left(\lim_{h\to 0^+} \frac{|h|}{h} = 1 \text{ and } \lim_{h\to 0^-} \frac{|h|}{h} = -1 \right)$$



differentiable at a point c roughly means "no sharp corners at c".

Fun fact: There are functions that are continuous everywhere but differentiable nowhere!

Theorem: Let f and g be functions differentiable at $x=c$ and $k \in \mathbb{R}$. Then $f \mp g$, $f \cdot g$, $k \cdot f$ and $\frac{f}{g}$ are differentiable at $x=c$
(for the differentiability of $\frac{f}{g}$, we need the condition that $g(c) \neq 0$.)
Moreover, whenever these maps are differentiable, we have that

- $(f \mp g)'(x) = f'(x) \mp g'(x)$
- $(k \cdot f)'(x) = k \cdot f'(x)$
- $(f \cdot g)'(x) = f'(x) g(x) + f(x) \cdot g'(x)$
- $\left(\frac{f}{g}\right)'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$

Proof (of the product rule):

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h} = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x+h) g(x) - f(x) g(x) + f(x+h) g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Example: Find an equation of the tangent line to the graph of $f(x) = \frac{3x^2+1}{x^2+1}$ at $x=2$

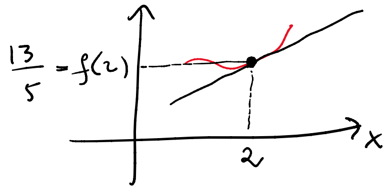
Solution: The slope of this tangent line is $f'(2)$.

$$f'(x) = \frac{d}{dx}\left(\frac{3x^2+1}{x^2+1}\right) = \frac{(3x^2+1)' \cdot (x^2+1) - (3x^2+1)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{((3x^2)' + 1')(x^2+1) - (3x^2+1)((x^2)' + 1')}{(x^2+1)^2}$$

$$= \frac{(3(x^2)' + 0)(x^2+1) - (3x^2+1)(2x+0)}{(x^2+1)^2} = \frac{6x(x^2+1) - 2x(3x^2+1)}{(x^2+1)^2}$$

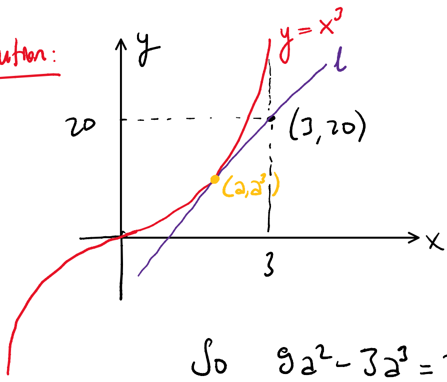
$f'(2) = \frac{8}{25}$. So an equation of the tangent line is



$$y - f(2) = f'(2)(x-2)$$
$$y - \frac{13}{5} = \frac{8}{25}(x-2)$$

Example: Find the line which is tangent to $y = x^3$ and passes through $(3, 20)$

Solution:



Let l be the line which is tangent to $y=x^3$ at (a, a^3). Then

$$m_l = \frac{d(x^3)}{dx}\bigg|_{x=a} = 3x^2\bigg|_{x=a} = 3a^2$$

$$= \frac{20 - a^3}{3 - a} = 3a^2$$

So $9a^2 - 3a^3 = 20 - a^3$. Thus $a = 2$

$$0 = 2a^3 - 9a^2 + 20$$

Therefore, an equation of l is $\boxed{y - 20 = \underbrace{12}_{m_l}(x-3)}$