18 February 2020

Tuesday, February 18, 2020 3:46 PM

Recall that f is continuous at c if $\lim_{x\to c} f(x) = f(c)$

Fact (not to be proved here): All the elementary functions that you'll learn in this course (i.e. poly's, rational functions, trig. functions, exp. and log functions, inverse trig functions) are continuous on their domain.

Theorem: Let f and g be defined on an interval containing a point c. If f and g are continuous at c, then so are

- $f \mp g$
- $f \cdot g$
- kf where $k \in \mathbb{R}$.
- f/g provided $g(c) \neq 0$

Proofs: Exercise. (These follow from the properties of limit.)

Theorem: Suppose that $(f\circ g)(x) = f(g(x))$ is defined on an interval containing a point c. If $\lim_{x\to c} g(x) = L$ exists and f is continuous at L, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

swap!

In particular, if f and g are continuous everywhere, then $f\circ g$ is continuous everywhere.

Example:

$$\lim_{x\to 1} \sqrt{\sin(x^2+1)} = \sqrt{\lim_{x\to 1} \sin(x^2+1)} = \sqrt{\sin\left(\lim_{x\to 1}(x^2+1)\right)}$$

$\sqrt{\cdot}$ is continuous sin is continuous

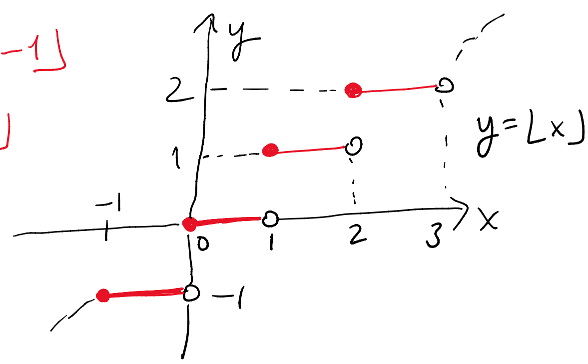
$$= \sqrt{\sin(1^2+1)} = \sqrt{\sin(2)}$$

x^2+1 is continuous

Example:

$$\lim_{x\to 1^-} \lfloor x-1 \rfloor = -1 \neq \lfloor 1-1 \rfloor$$

$$\neq \left\lfloor \lim_{x\to 1^-} x-1 \right\rfloor$$



Example: Find constant a and b such that the function

$$f(x) = \begin{cases} x^2+a & \text{if } x>1 \\ 5 & \text{if } x=1 \\ \sqrt{x^2+b} & \text{if } x<1 \end{cases}$$

is continuous everywhere.

Solution: If $c>1$ or $c<1$, then f is identically x^2+a or $\sqrt{x^2+b}$ on a neighborhood of c and so is continuous at c. Thus, it suffices to guarantee the continuity of f at $c=1$. So we need to choose a and b such that $\lim_{x\to 1} f(x) = f(1)$

$$\left.\begin{aligned} &\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+a = 1^2+a = a+1 \\ &\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x^2+b} = \sqrt{1+b} \\ &f(1) = 5 \end{aligned}\right\}$$

We want to have

$a+1 = \sqrt{1+b} = 5$

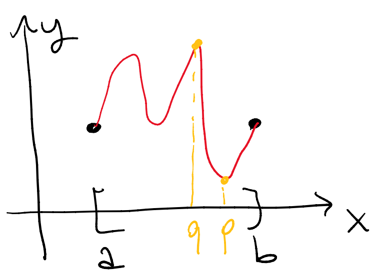
Choose $a=4$ and $b=24$. Then we have

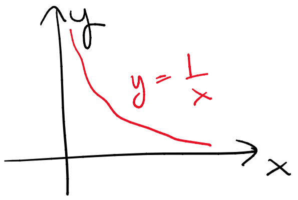
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$ and so f becomes continuous at 1.

Two important theorems about continuous functions

The max-min theorem: Let f be a continuous function defined on a closed and bounded interval $[a,b]$. Then there are points p and q in $[a,b]$ such that

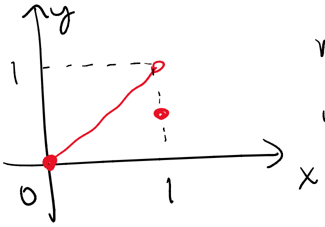
$$f(p) \le f(x) \le f(q) \quad \text{for all } x \text{ in } [a,b].$$





no maximum on $(0,\infty)$

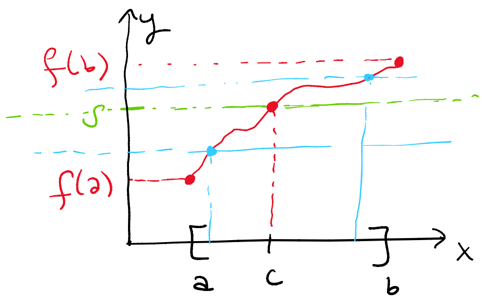
no minimum on $(0,\infty)$

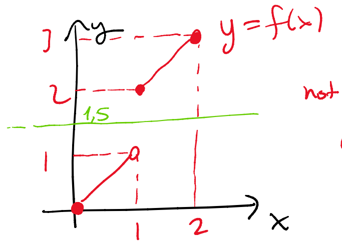


minimum at 0

no maximum on $[0,1]$

Intermediate Value Theorem: Let f be continuous on a closed and bounded interval $[a,b]$. For any s between $f(a)$ and $f(b)$, there exists some c in $[a,b]$ such that $f(c)=s$.



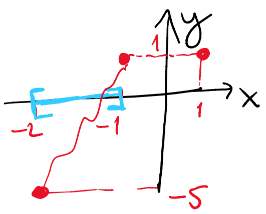


$y=f(x)$

not continuous at $x=1$

Example: Show that the equation $x^3-x+1=0$ has a root.

Solution: Let $f(x)=x^3-x+1$. Then f is continuous everywhere and so is continuous on $[-2,1]$. Since



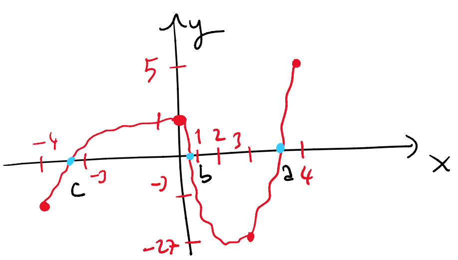
$$f(-2)=-5<\overset{s}{0}<1=f(1),$$

by IVT, there is c in $[-2,-1]$ such that $f(c)=0$, so

$$c^3-c+1=0$$

Example: Show that the equation $x^3-15x+1=0$ has at least three solutions on $[-4,4]$.

Solution:



Let $f(x)=x^3-15x+1$. Then f is continuous everywhere.

Since $f(3)<0<f(4)$

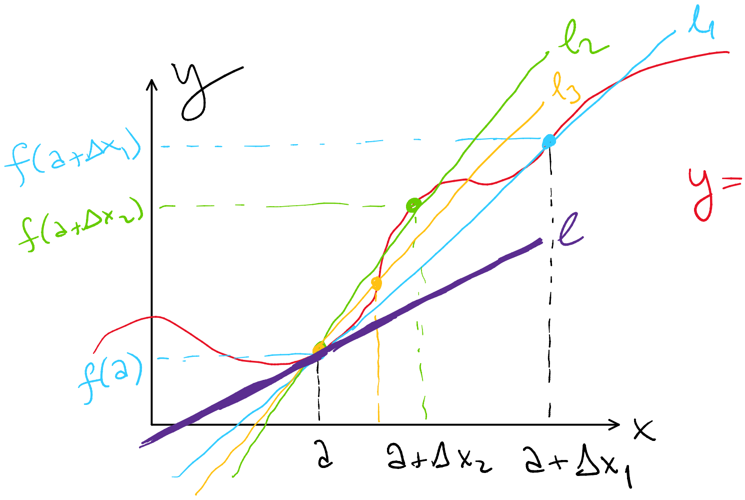
and $f(1)<0<f(0)$

and $f(-4)<0<f(-3)$,

by IVT, there is a in $[3,4]$ and b in $[0,1]$ and c in $[-4,-3]$ such that $f(a)=f(b)=f(c)=0$.

For further use of IVT in a smart way, Google "Bisection method".

Differentiation



$$m_{\ell_1} = \frac{f(a+\Delta x_1) - f(a)}{\Delta x_1}$$

$$m_{\ell_2} = \frac{f(a+\Delta x_2) - f(a)}{\Delta x_2}$$

Defn: Let f be defined on an interval containing a point c. The derivative of f at c is the limit

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

$\rightsquigarrow$ the slope of the secant line through $(c, f(c))$ and $(c+h, f(c+h))$

If this limit exists, we say that f is differentiable at c.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}}$$