13 February 2020

13 Şubat 2020 Perşembe 12:30

Example: Find $\lim\limits_{x\to-\infty} \sqrt{x^2+2x}+x$

Solution:

$$\lim_{x\to-\infty}\left(\sqrt{x^2+2x}+x\right) = \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+2x}+x\right)\left(\sqrt{x^2+2x}-x\right)}{\left(\sqrt{x^2+2x}-x\right)}$$

$$= \lim_{x\to-\infty} \frac{(\cancel{x^2}+2x)-\cancel{x^2}}{\sqrt{x^2+2x}-x} = \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+2x}-x}$$

$$= \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2\left(1+\frac{2}{x}\right)}-x}$$

$\sqrt{x^2}=|x|$

$$= \lim_{x\to-\infty}\frac{2x}{|x|\sqrt{1+\frac{2}{x}}-x}$$

$$= \lim_{x\to-\infty}\frac{2x}{-x\sqrt{1+\frac{2}{x}}-x} = \lim_{x\to-\infty}\frac{2\cancel{x}}{-\cancel{x}\left(\sqrt{1+\frac{2}{x}}+1\right)}$$

$$= \lim_{x\to-\infty}\frac{2}{(-1)\cdot\left(\sqrt{1+\cancel{\tfrac{2}{x}}^{\,0}}+1\right)}$$

$$= \frac{2}{(-1)\cdot(1+1)} = -1$$

An unjustified / wrong solution:

$$\lim_{x\to-\infty}\sqrt{(x+1)^2-1}+x = \lim_{x\to-\infty}\sqrt{(x+1)^2\left(1-\frac{1}{(x+1)^2}\right)}+x$$

$$= \lim_{x\to-\infty} |x+1|\cdot\sqrt{1-\cancel{\frac{1}{(x+1)^2}}^{\,0}}+x$$

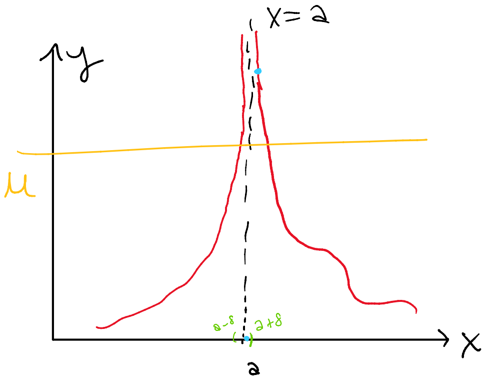
???

$$= \lim_{x\to-\infty}|x+1|+x = \lim_{x\to-\infty} -(x+1)+x$$

$$= \lim_{x\to-\infty} -x-1+x = -1$$

WRONG!!!

Infinite limits



An informal defn: Let f be a function defined on a neighborhood of a point a, possibly except at a.

We write $\lim\limits_{x\to a} f(x)=+\infty$

if $f(x)$ values can be made arbitrarily large by taking x values sufficiently close to a.

Formal defn: For all M, there is $\delta>0$ such that if $0<|x-a|<\delta$, then $f(x)>M$

$\Updownarrow$

$$\lim_{x\to a} f(x)=+\infty$$

One can similarly define the notions

$$\lim_{x\to a} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = \mp\infty, \quad \lim_{x\to a^-} f(x) = \mp\infty$$

Terminological warning: If $\lim_{x\to a} f(x) = \mp\infty$, then we still say that "f does not have a limit at a", that is, when we say that "f has a limit at a", it means that $\lim_{x\to a} f(x)$ is in $\mathbb{R}$.

Examples:

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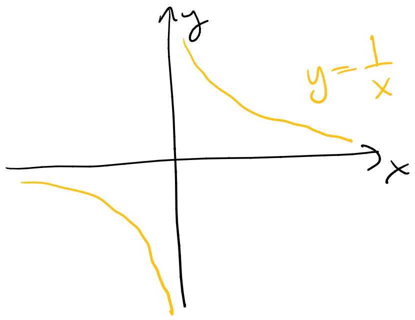
$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$

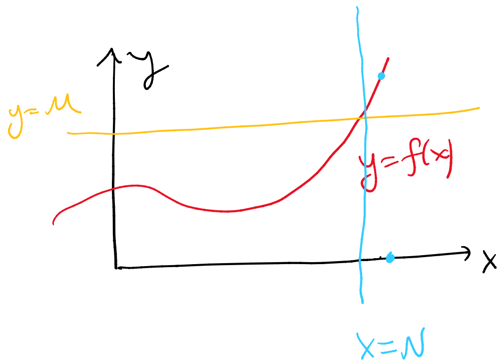
- 

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

(so $\lim_{x\to 0} \frac{1}{x} \neq +\infty$)

An informal defn: We write $\lim_{x\to \infty} f(x) = +\infty$ if $f(x)$ values can be made arbitrarily large by taking x values arbitrarily large

(Formal defn: For all M, there exists N such that if $x > N$, then $f(x) > M$)



One can similarly define $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \mp\infty$

Examples:

- $\lim_{x\to\infty} -2x^2+x+4 = -\infty$
- $\lim_{x\to-\infty} x^4+x-2 = +\infty$
- $\lim_{x\to-\infty} -2x^3+x^2 = +\infty$

Example: Find $\lim_{x\to0^+} \frac{x^2-3x-10}{-x^2+5x}$

Solution:

$$\lim_{x\to0^+} \frac{x^2-3x-10}{-x^2+5x} = \lim_{x\to0^+} \frac{\cancel{(x-5)}(x+2)}{-\cancel{(x-5)}x} = \lim_{x\to0^+} -\frac{x+2}{x}$$

$$= \lim_{x\to0^+} -1-\frac{2}{x} = -\infty$$

Fact:

depends on whether $x\to+\infty$ or $x\to-\infty$, and, the coefficients a_n, b_m and n and m

$$\lim_{x\to\mp\infty} \frac{a_0+a_1x+a_2x^2+\dots+a_nx^n}{b_0+b_1x+b_2x^2+\dots+b_mx^m} = \begin{cases} 0 & \text{if } n<m \\ \mp\infty & \text{if } n>m \\ \frac{a_n}{b_m} & \text{if } n=m \end{cases}$$

Examples:

- $\lim_{x\to-\infty} \frac{\frac{1}{x^4}(1+x+x^3)}{\frac{1}{x^4}(7-x^4)} = \lim_{x\to-\infty} \frac{\cancel{\frac{1}{x^4}}^{0}+\cancel{\frac{1}{x^3}}^{0}+\cancel{\frac{1}{x}}^{0}}{\left(\cancel{\frac{7}{x^4}}^{0}-1\right)} = \frac{0+0+0}{0-1} = 0$

- $\lim_{x\to-\infty} \frac{7+x-2x^3}{x+x^2} = \lim_{x\to-\infty} \frac{\frac{7}{x^3}+\frac{1}{x^2}-2}{\frac{1}{x^2}+\frac{1}{x}} = +\infty$

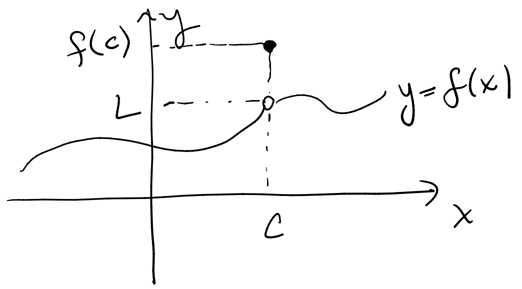
- $$\lim_{x\to-\infty} \frac{\frac{1}{x^2}(2-x-x^2)}{\frac{1}{x^2}(1+x^2)} = \lim_{x\to-\infty} \frac{\cancel{\frac{2}{x^2}}^{\to 0} - \cancel{\frac{1}{x}}^{\to 0} - 1}{\cancel{\frac{1}{x^2}}^{\to 0} + 1} = \frac{0-0-1}{0+1} = -1$$

Continuity

Defn: let f be a function defined on some interval (a,b) and let c be in (a,b). We say that f is continuous at c if $\boxed{\lim_{x\to c} f(x) = f(c)}$

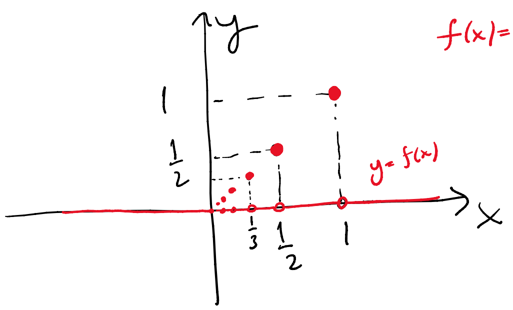
- When c is a right-end point of the domain of f, we say that f is continuous at c if $\lim_{x\to c^-} f(x) = f(c)$. Similarly f is said to be continuous at a left-end point c of its domain if $\lim_{x\to c^+} f(x) = f(c)$

We say that f is continuous if f is continuous at every point in its domain.

- 

$$\lim_{x\to c} f(x) \neq f(c)$$

f is not continuous at c

- 

$$f(x) = \begin{cases} x & \text{if } x = \frac{1}{n} \text{ for a positive integer } n \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

f is continuous at 0