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Indeterminate forms

Suppose that we are (means taking limits) trying to evaluate a limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ where $a = 0, \mp\infty$. When we try to "substitute" a in $f(x)$ and $g(x)$, we sometimes get $\frac{0}{0}$ or $\frac{\mp\infty}{\mp\infty}$. In these cases, we say that $\frac{f(x)}{g(x)}$ has an indeterminate form at $x=a$ (of the relevant type.) There are other types of indeterminate forms which can be turned into these forms.

	types	examples
L'H	$\frac{0}{0}$	$\lim_{x\to 0}\frac{\sin x}{x}$
L'H	$\frac{\infty}{\infty}$	$\lim_{x\to 0}\frac{\frac{1}{x^2}}{e^{\frac{1}{x^2}}}$
	$0.\infty$	$\lim_{x\to 0^+} x.\ln\left(\frac{1}{x}\right)$
	$\infty-\infty$	$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$
	0^0	$\lim_{x\to 0^+} x^x$
	∞^0	$\lim_{x\to\frac{\pi}{2}^-}(\sec x)^{\cos x}$
	1^∞	$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

L'Hopital's rule: let f and g be differentiable on (a,b) and $g'(x)\neq 0$ on (a,b).

- If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ and $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$ (where $-\infty \leq L \leq +\infty$), then $\lim_{x\to a^+}\frac{f(x)}{g(x)} = L$.

  equality allowed

- If ______________ $= \mp\infty$ ______________________ , ______________ .

These results also hold for the case $\lim_{x\to b^-}$ and $\lim_{x\to c}$ with $a<c<b$. The cases $a=-\infty$ or $b=+\infty$ are also allowed.

Example: $\lim_{x\to 0}\frac{2\sin x - x}{e^x - 1} = 1$

$$\lim_{x\to 0}\frac{2\sin x - x}{e^x - 1} \overset{L'H}{\underset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 0}\frac{2\cos x - 1}{e^x} = \frac{2-1}{1} = 1$$

Example: $\lim_{x\to 0}\frac{\frac{1}{x^2}}{e^{\frac{1}{x^2}}} = 0$

$$\lim_{x\to 0}\frac{\frac{1}{x^2}}{e^{\frac{1}{x^2}}} \overset{L'H}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to 0}\frac{\cancel{\frac{-2}{x^3}}}{e^{\frac{1}{x^2}}\cdot\cancel{\frac{-2}{x^3}}} = \lim_{x\to 0} e^{\frac{-1}{x^2}} = 0$$

Example: $\lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\tan x}{\sec x} = \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\frac{\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}} = \lim\limits_{x\to\frac{\pi}{2}^-} \sin x = 1$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} \overset{L'H}{\left(\frac{+\infty}{+\infty}\right)} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^{\cancel{2}} x}{\cancel{\sec x}\tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan x} \overset{L'H}{\left(\frac{+\infty}{+\infty}\right)} \lim_{x\to\frac{\pi}{2}^-} \frac{\cancel{\sec x}\tan x}{\sec^{\cancel{2}} x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} \overset{L'H}{\cdots}$$

$\boxed{\text{Moral of the story: Do not blindly apply L'Hôpital's rule for the process may never stop.}}$

$\boxed{\text{WARNING: Do not apply L'Hôpital's rule unless you have an indeterminate type.}}$

L'Hôpital's requires us to have indeterminate types of the form $\frac{0}{0}$ and $\frac{\mp\infty}{\mp\infty}$. What do we do if we have other types of indeterminacies?

$0\cdot\infty \rightsquigarrow$ rewrite using the reciprocal of one of the factors $\rightsquigarrow$ you'll get $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example: $\lim\limits_{x\to 0^+} x\cdot \ln\left(\frac{1}{x}\right) = \lim\limits_{x\to 0^+} \dfrac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}} = 0$

$$\lim_{x\to 0^+} \frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{\left(\frac{\infty}{\infty}\right)} \lim_{x\to 0^+} \frac{\frac{1}{\frac{1}{x}}\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = 0$$

$$\lim_{x\to 0^+} \frac{x}{\frac{1}{\ln\left(\frac{1}{x}\right)}} \overset{L'H}{} \lim_{x\to 0^+} \frac{1}{\frac{-1}{\ln^2\left(\frac{1}{x}\right)}\cdot\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{\ln^2\left(\frac{1}{x}\right)} = 0$$

$\infty - \infty \rightsquigarrow$ execute algebraic operations (e.g. divide and multiply conjugate, cross-multiply) $\rightsquigarrow$ you'll get $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example: $\lim\limits_{x\to 0^+} \dfrac{1}{x} - \dfrac{1}{\sin x} = 0$

$$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} \overset{L'H}{\left(\frac{0}{0}\right)} \lim_{x\to 0^+} \frac{\cos x - 1}{1.\sin x + x.\cos x} \overset{L'H}{\left(\frac{0}{0}\right)} \lim_{x\to 0^+} \frac{-\sin x}{\cos x + (1.\cos x + x(-\sin x))} = 0$$

0^0, ∞^0, 1^∞ $\rightsquigarrow$ rewrite $f(x)^{g(x)}$ as $e^{g(x)\ln f(x)}$ and take the limit of $\underbrace{g(x)\ln f(x)}_{0.\mp\infty}$ (using L'H) and move the limit to exponent

Example: $\lim\limits_{x\to 0^+} x^x = 1$

the exponential function is continuous at 0

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1$$

$$\boxed{\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0}$$

Example: $\lim\limits_{x\to\frac{\pi}{2}^-} \sec x^{\cos x} = \lim\limits_{x\to\frac{\pi}{2}^-} e^{\cos x\,\ln\sec x} = e^{\lim_{x\to\frac{\pi}{2}^-} \cos x\,\ln\sec x} = e^0 = 1$

$$\boxed{\lim_{x\to\frac{\pi}{2}^-} \cos x\,\ln(\sec x) = \lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\sec x)}{\sec x} \overset{L'H}{\left(\frac{\infty}{\infty}\right)} \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\sec x}\cdot\cancel{\sec x\tan x}}{\cancel{\sec x\tan x}} = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\sec x} = \lim_{x\to\frac{\pi}{2}^-} \cos x = 0}$$

Example: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})} = e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} = e^1 = e$

$$\boxed{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} \underset{\left(\frac{0}{0}\right)}{\overset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = 1}$$

Examples for you to be careful:

- $\lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by squeeze theorem

$$\lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{x} \overset{L'H}{=} \lim_{x\to 0}\frac{2x\sin(\frac{1}{x}) + \cancel{x^2}\cdot\cos(\frac{1}{x})\cdot\frac{-1}{\cancel{x^2}}}{1} = \lim_{x\to 0} \overset{\to 0}{\cancel{2x\sin\left(\frac{1}{x}\right)}} - \cos\left(\frac{1}{x}\right) \quad \text{d.n.e}$$

Moral of the story: If after applying L'Hopital's rule you get a limit which does not exist, then this means nothing!

- $\lim_{x\to\infty}\frac{(x+\cancel{\sin x\cos x})}{(x+\cancel{\sin x\cos x})\cdot e^{\sin x}} = \lim_{x\to\infty} e^{-\sin x}$ d.n.e

The condition that $g'(x)\neq 0$ in L'Hopital not satisfied

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{L'H}{=}} \lim_{x\to\infty}\frac{1+\cos^2x-\sin^2x}{(1+\cos^2x-\sin^2x)\cdot e^{\sin x}+(x+\sin x\cos x)\,e^{\sin x}\cdot\cos x} = \lim_{x\to\infty}\frac{2\cos^{\cancel{2}}x}{2\cos^{\cancel{2}}x\cdot e^{\sin x}+(x+\sin x\cos x)\,e^{\sin x}\,\cancel{\cos x}}$$

$$= \lim_{x\to\infty}\frac{2\cos x}{e^{\sin x}(2\cos x + x + \sin x\cos x)} = 0$$