11 February 2020

13 Şubat 2020 Perşembe 11:08

Recall that $\lim_{x\to a} f(x) = L$ iff ($\Longleftrightarrow$) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

- One can make such formal definitions (using ε-δ) for one-sided limits as well.

$\lim_{x\to a^+} f(x) = L$ iff For every $\varepsilon > 0$ there is $\delta > 0$ with if $0 < x-a < \delta$ then $|f(x) - L| < \varepsilon$

$(x \to a^-)$ $\quad 0 < a-x < \delta$

Some properties of limit Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

- $\lim_{x\to a^{\mp}} (f(x) \mp g(x)) = \lim_{x\to a^{\mp}} f(x) \mp \lim_{x\to a^{\mp}} g(x)$
- $\lim_{x\to a^{\mp}} (f(x) \cdot g(x)) = \left(\lim_{x\to a^{\mp}} f(x)\right) \cdot \left(\lim_{x\to a^{\mp}} g(x)\right)$
- $\lim_{x\to a^{\mp}} (c f(x)) = c\left(\lim_{x\to a^{\mp}} f(x)\right)$ for all $c \in \mathbb{R}$
- $\lim_{x\to a^{\mp}} \left(\frac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to a^{\mp}} f(x)}{\lim_{x\to a^{\mp}} g(x)}$ provided that $\lim_{x\to a^{\mp}} g(x) \neq 0$

WARNING: The assumptions that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist are necessary. For example,

$$5 = \lim_{x\to 0} 5 = \lim_{x\to 0}\left(\left(\frac{1}{x}+5\right) - \frac{1}{x}\right) \neq \lim_{x\to 0} \frac{1}{x} + 5 - \lim_{x\to 0} \frac{1}{x}$$

Proof of the first item: Suppose that $\lim_{x\to a} f(x) = K$ and $\lim_{x\to a} g(x) = L$

We wish to show $\lim_{x\to a} (f(x) + g(x)) = K + L$

Given $\varepsilon > 0$.

- Because $\lim_{x\to a} f(x) = K$, for $\varepsilon_1 = \frac{\varepsilon}{4}$, there is $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$, then $|f(x) - K| < \varepsilon_1 = \frac{\varepsilon}{4}$
- Similarly, because $\lim_{x\to a} g(x) = L$, for $\varepsilon_2 = \frac{\varepsilon}{4}$, there is $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|g(x) - L| < \varepsilon_2 = \frac{\varepsilon}{4}$

Choose $\delta = \min\{\delta_1, \delta_2\} > 0$. Then

if $0 < |x-a| < \delta$ ($\leq \delta_1$, $\leq \delta_2$), then $|f(x) + g(x) - (K+L)| = |f(x) - K + g(x) - L|$

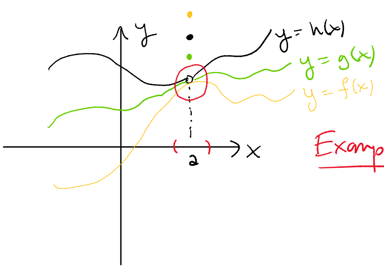
$$\leq |f(x) - K| + |g(x) - L| = \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$$

So $\lim_{x\to a} f(x) + g(x) = K + L$

- The other properties can be proven similarly.
- These limit properties also hold for one-sided limits.

The Squeeze Theorem: Suppose that the inequality $f(x) \leq g(x) \leq h(x)$ holds on a neighborhood of a, possibly except at a. Then

If $\lim_{x\to a^{\mp}} f(x) = \lim_{x\to a^{\mp}} h(x) = L$, then $\lim_{x\to a^{\mp}} g(x) = L$.



- Similar statements hold for one sided limits as well.

Example: Show that $\lim_{x\to 0} x\left(\sin\left(\frac{1}{x}\right) + \cos^2(x)\right) = 0$

Solution: For all $x \neq 0$, we have

$$|\sin(\tfrac{1}{x}) + \cos^2(x)| \leq |\sin(\tfrac{1}{x})| + |\cos^2(x)| \leq 1 + 1 = 2$$

So $0 \leq |x(\sin(\frac{1}{x}) + \cos^2(x))| \leq 2|x|$ for all $x \neq 0$.

Since $\lim_{x\to 0} 0 = \lim_{x \to 0} 2|x| = 0$, by the squeeze theorem

$\lim_{x\to 0} |x(\sin(\frac{1}{x}) + \cos^2(x))| = 0$ and so $\lim_{x \to 0} x(\sin(\frac{1}{x}) + \cos^2(x)) = 0$.

Fact: If $\lim_{x\to a} |f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$

Proof: We have that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in \mathbb{R}$.

As $\lim_{x\to a} |f(x)| = \lim_{x \to a} -|f(x)| = 0$, by squeeze theorem

$\lim_{x\to a} f(x) = 0$.

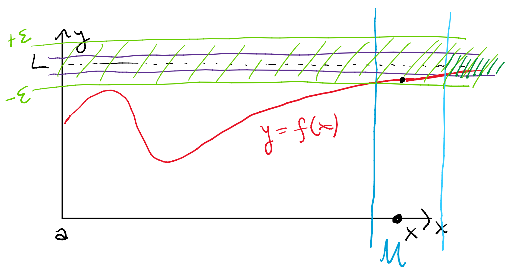
Example: Show that $\lim_{x\to 0^+} \sqrt{x \cdot |\sin(\frac{1}{x})|} = 0$

Solution: We have that $0 \leq \sqrt{x\,|\sin(\frac{1}{x})|} \leq \sqrt{x}$ for $x \geq 0$.

Moreover, $\lim_{x\to 0^+} 0 = \lim_{x \to 0^+} \sqrt{x} = 0$ and so by the squeeze theorem $\lim_{x\to 0^+} \sqrt{x|\sin(\frac{1}{x})|} = 0$.

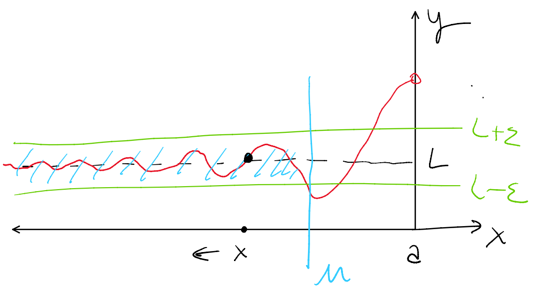
will be explained later while covering continuity

Limits at infinity and infinite limits



Informal defn: Let f be a function defined on an interval (a, ∞) (or $(-\infty, a)$). We say that the limit of f as x approaches $+\infty$ (or $-\infty$) is L and write $\lim_{x\to\infty} f(x) = L$ (or $\lim_{x\to -\infty} f(x) = L$) if $f(x)$ values can be made arbitrarily close to L by taking large enough (or, small enough) x values.

More formally, $\lim_{x\to\infty} f(x) = L$ iff For every $\varepsilon > 0$ there is M such that if $x > M$ then $|f(x) - L| < \varepsilon$



$\lim_{x\to -\infty} f(x) = L$ iff For every $\varepsilon > 0$ there is M such that if $x < M$ then $|f(x) - L| < \varepsilon$

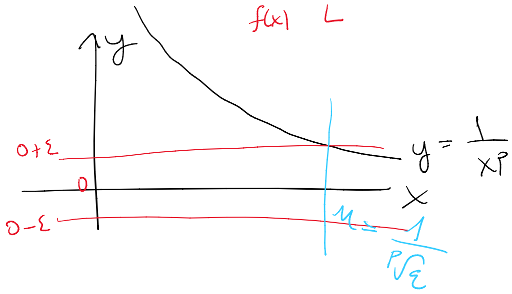
Example: Suppose that $p>0$. Show that $\lim_{x\to\infty} \frac{1}{x^p}=0$

Solution: Given $\varepsilon>0$, choose $M=\frac{1}{\sqrt[p]{\varepsilon}}>0$. Then

if $x>M^{>0}$, then $x>\frac{1}{\sqrt[p]{\varepsilon}}$ and so $\sqrt[p]{\varepsilon}>\frac{1}{x}$

$$\varepsilon>\frac{1}{x^p}>0$$

which means $\left|\underbrace{\frac{1}{x^p}}_{f(x)}-\underbrace{0}_{L}\right|<\varepsilon$



Example: Find $\lim_{x\to\infty}\frac{\sin x+x}{\cos\left(\frac{1}{x}\right)+x^2}$

Solution:
$$\lim_{x\to\infty}\frac{(\sin x+x)\cdot\frac{1}{x^2}}{\left(\cos\left(\frac{1}{x}\right)+x^2\right)\cdot\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\overset{\to 0\,\star}{\frac{\sin x}{x^2}}+\overset{\to 0}{\frac{1}{x}}}{\underset{\to 0}{\frac{\cos\left(\frac{1}{x}\right)}{x^2}}+\underset{\to 1}{1}}=\frac{0+0}{0+1}=0$$

In order to say this, we needed to check that the limits $\lim_{x\to\infty}\frac{\sin x}{x^2}$, $\lim_{x\to\infty}\frac{1}{x}$, $\lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)}{x^2}$ and $\lim_{x\to\infty}1$ all separately exist.

Show that this limit is 0 by squeeze theorem

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$$\frac{-1}{x^2}\le\frac{\sin x}{x^2}\le\frac{1}{x^2}\quad\text{for all } x\neq 0$$

As $\lim_{x\to\infty}\frac{1}{x^2}=\lim_{x\to\infty}\frac{-1}{x^2}=0$, by squeeze theorem, $\lim_{x\to\infty}\frac{\sin x}{x^2}=0$

Example: Find $\lim_{x\to-\infty}\frac{\sqrt{x^2+x}+\sin x}{x+5}$

Solution:
$$\lim_{x\to-\infty}\frac{\left(\sqrt{x^2+x}+\sin x\right)\cdot\frac{1}{x}}{(x+5)\cdot\frac{1}{x}}=\lim_{x\to-\infty}\frac{\overset{\to -1}{\frac{\sqrt{x^2+x}}{x}}+\overset{\to 0 \text{ by squeeze theorem}}{\frac{\sin x}{x}}}{\underset{\to 1}{1}+\underset{\to 0}{\frac{5}{x}}}=-1$$

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+x}}{x}=\lim_{x\to-\infty}\frac{\sqrt{x^2\left(1+\frac{1}{x}\right)}}{x}$$
$$=\lim_{x\to-\infty}\frac{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x}}}{x}$$
$$=\lim_{x\to-\infty}\frac{|x|\cdot\sqrt{1+\frac{1}{x}}}{x}=\lim_{x\to-\infty}\frac{-x\cdot\sqrt{1+\frac{1}{x}}}{x}=\lim_{x\to\infty}-\sqrt{1+\frac{1}{x}}=-1$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2(1+\frac{1}{x})}}{x}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x}}}{x}$$

$$= \lim_{x\to-\infty} \frac{|x|\cdot\sqrt{1+\frac{1}{x}}}{x} = \lim_{x\to-\infty} \frac{-x\cdot\sqrt{1+\frac{1}{x}}}{x} = \lim_{x\to\infty} -\sqrt{1+\frac{1}{x}} = -1$$

WARNING: $$\lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+x}}{x} + \frac{\sin x}{x}}{1+\frac{5}{x}} = \cancel{\lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+x}}{x}+0}{1+0}}$$

YOU CAN'T TAKE PARTIAL LIMITS LIKE THIS! HERE THAT YOU GET THE CORRECT ANSWER IS AN ACCIDENT!