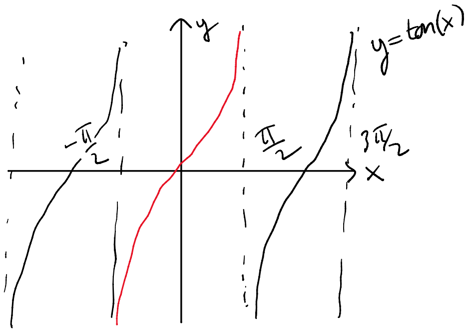
10 March 2020

Tuesday, March 10, 2020 3:42 PM

Recall that we've introduced the inverse maps for sine and cosine last time. We shall next see the inverse of tangent.



Although $\tan(x)$ is not one-to-one in its usual domain, if we restrict it to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ becomes one-to-one and so has an inverse map

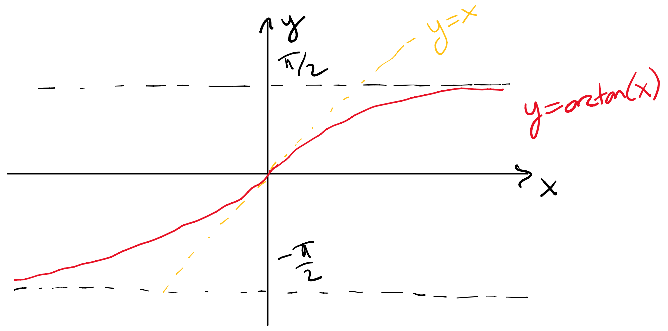
$$\arctan: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This map is called the arctangent map and satisfies

$$\tan(\arctan(x)) = x \quad \text{for all } x \in \mathbb{R}$$

$\arctan\left(\tan\left(\frac{9\pi}{4}\right)\right) = \frac{\pi}{4}$ WARNING

$$\arctan(\tan(x)) = x \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$



The derivative of arctan is given by

$$\tan(\arctan(x)) = x$$

$$\left(1 + \tan^2(\arctan(x))\right) \cdot \frac{d}{dx} \arctan x = 1$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Example: Show that $\arctan x < x$ for all $x > 0$.

Solution: Let $f(x) = x - \arctan x$. Then $f'(x) = 1 - \frac{1}{1+x^2} > 0$ for all x. (<1 if $x \neq 0$)

Thus f is increasing on $(-\infty, \infty)$. Also $f(0) = 0 - \arctan 0 = 0$. Thus, for any $x > 0$, $f(x) > f(0) = 0$. Therefore, for any $x > 0$, $x - \arctan x > 0$

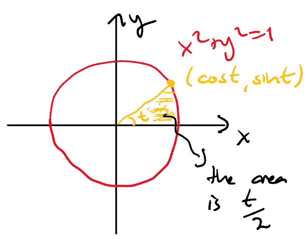
x		0	
f'	$+$		$+$
f	$\nearrow$		$\nearrow$

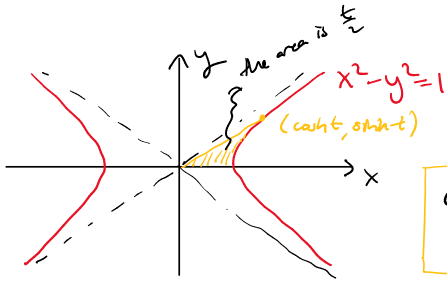
Read about the inverse of secant from your book! We shall not cover it here.

$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$

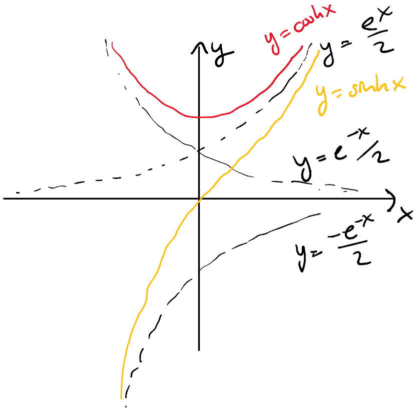
Hyperbolic functions We define the hyperbolic sine and cosine as follows.

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$





$$\boxed{\begin{aligned} \cosh^2 t - \sinh^2 t &= 1 \\ \cos^2 t + \sin^2 t &= 1 \end{aligned}}$$



$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

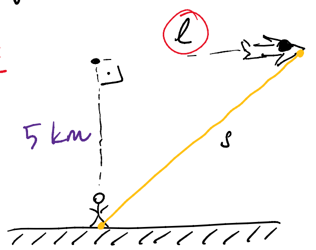
One can also define hyperbolic tangent, secant etc.

Related rates

Suppose that we have two or more quantities all of which depend on the same variable (or differentiable maps.) If one can relate these quantities by some equations, then, by differentiating both sides of these equations, one can relate the rates of changes of these quantities to each other.

Example: An aircraft is flying horizontally at a speed of 400 km/h at an altitude of 5 km. How fast is the distance between an observer and the aircraft changing after 30 minutes passes from the moment at which the aircraft was directly above the observer?

Solution:



By Pythagorean theorem,

$$s^2 = \ell^2 + 25$$

and so we have $2s\frac{ds}{dt} = 2\ell \frac{d\ell}{dt} + 0$

400 km/h

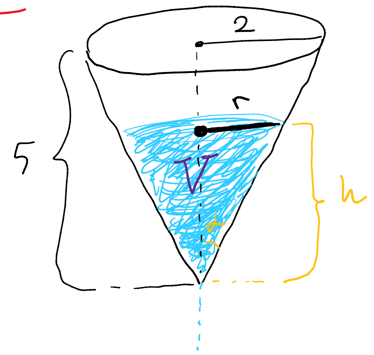
We wish to find $\frac{ds}{dt}$ at the moment when $\ell = 200$ km (and so, $s = \sqrt{200^2+5^2}$ km.) Thus $\cancel{2}\cdot\sqrt{200^2+5^2}\,\frac{ds}{dt} = \cancel{2}\cdot 200 \cdot 400$

and so $\frac{ds}{dt} = \frac{80000}{\sqrt{200^2+5^2}}$

Example: A leaky water tank is in the shape of an inverted right circular cone with depth 5m and top radius 2m. When the water in the tank is 4 m deep, it is leaking from the bottom at a rate of $\frac{1}{4}$ m^3/sec. How fast is the water level dropping at this moment?

the rate of change of the volume

Solution:



We have

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{1}{3}\pi\frac{4}{25}h^3$$

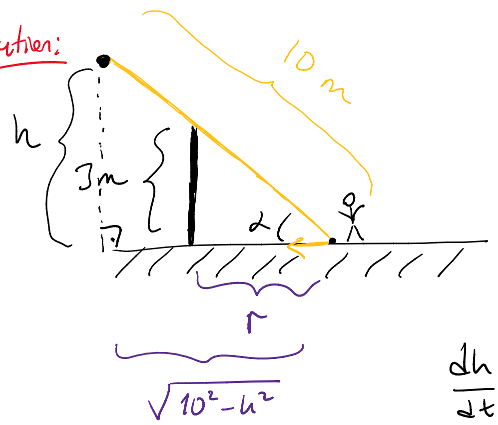
$$\tan\alpha = \frac{r}{h} = \frac{2}{5}$$

Differentiating both sides gives us

$$\frac{dV}{dt} = \frac{4\pi}{75}3h^2\frac{dh}{dt}.$$ We have that $\frac{dV}{dt} = \frac{-1}{4}$ (because V is decreasing) and so when $h=4$ we have $$\frac{-1}{4} = \frac{4\pi}{75}\cdot 3\cdot 4^2\cdot\frac{dh}{dt}$$ and thus $$\frac{dh}{dt} = \frac{-75}{\pi\, 3.4^4}$$ (This indicates that h is decreasing)

Example: A 10 m-ladder is hanging over a fence of 3 m while its other end is on the ground. If one pushes the end on the ground towards the fence at a rate of $\frac{1}{2}$ m/sec, how fast does the height of the tip of the ladder hanging over the fence change when the end of the ladder on the ground is 4 m away from the fence?

Solution:



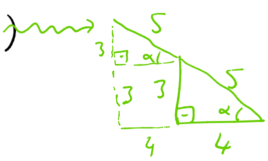
By Pythagorean theorem and similarity, we have

$$\frac{h}{\sqrt{10^2-h^2}} = \frac{3}{r} \; (=\tan\alpha)$$

and so $hr = 3\sqrt{10^2-h^2}$. Differentiating this gives us,

$$\frac{dh}{dt}\cdot r + h\frac{dr}{dt} = 3\,\frac{1}{2\sqrt{10^2-h^2}}\cdot(-2h)\cdot\frac{dh}{dt}$$

As we have, $\frac{dr}{dt} = \frac{-1}{2}$, when $r=4$ (and so $h=6$)



we get $$\frac{dh}{dt}\cdot 4 + 6\cdot\left(\frac{-1}{2}\right) = \frac{3}{2\sqrt{10^2-6^2}}\cdot(-2.6)\,\frac{dh}{dt}$$

$$4\frac{dh}{dt} - 3 = \frac{3}{\cancel{2.8}\,4}\cdot\cancel{(-12)}^{\cancel{-6}\,-3}\,\frac{dh}{dt}$$ and so $$\frac{dh}{dt} = \frac{3}{4+\frac{9}{4}} = \frac{12}{25}$$