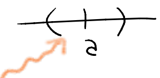
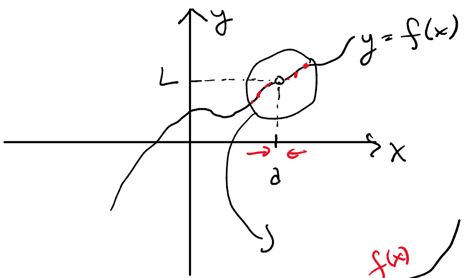
06 February 2020

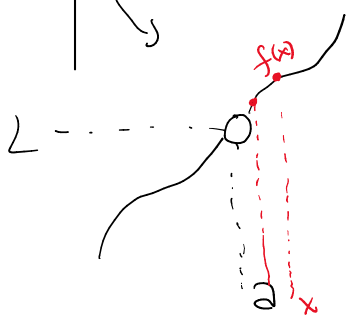
Limit of a function



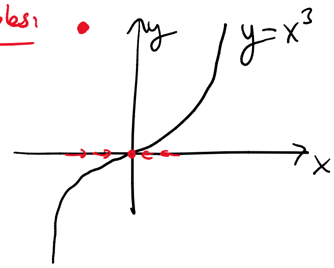
An informal definition: Let f be a function defined for all x values in a neighborhood of a point a (possibly except at a). We say that the limit of f at a is L if "the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a." In this case we write

$$\lim_{x \to a} f(x) = L$$



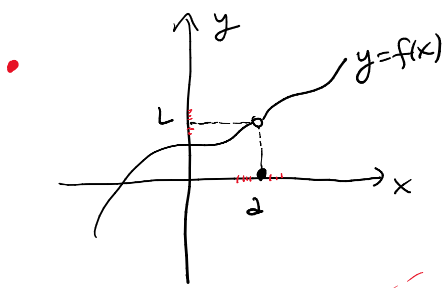


Examples:

- 

x	0.1	0.01	0.001	......
$f(x)$	0.001	0.000001	0.000000001	...

$$\lim_{x \to 0} x^3 = 0$$

- 

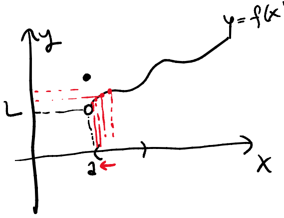
$$\lim_{x \to a} f(x) = L \neq f(a)$$

Fact: Assume for now that the limits of "elementary functions" (poly., rational, trig., log. etc.) at a point can be found by plugging in that point whenever the function is defined at that point. (We shall learn why we can do this later.)

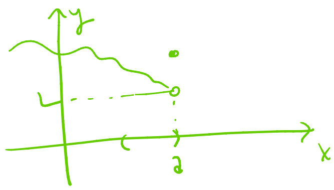
Example:
$$\lim_{x \to 2} \frac{x^2+3}{x^3+7} = \frac{2^2+3}{2^3+7} = \frac{7}{15}$$

$$\lim_{x \to 3} \frac{x^2-9}{x^3-27} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x^2+3x+9)} = \lim_{x \to 3} \frac{x+3}{x^2+3x+9} = \frac{3+3}{9+9+9} = \frac{6}{27}$$

One-sided limits

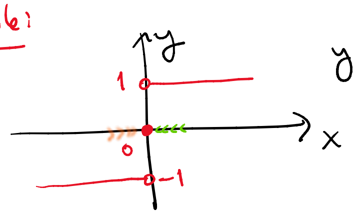


(An informal definition) Let f be a function which is defined on some open interval (a, b) ((b, a)). We say that the right-limit (left-limit) of f at a is L if $f(x)$ values can be made arbitrarily close to L by taking x values greater than (less than) a (i.e. on the right side (left-side) of a) sufficiently close to a. In which case we write



$$\lim_{x\to a^+} f(x) = L \qquad \left(\lim_{x\to a^-} f(x) = L\right)$$

Example:

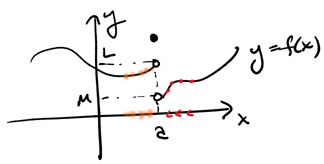


$$y = \operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^+} \operatorname{sgn}(x) = 1 \neq \lim_{x\to 0^-} \operatorname{sgn}(x) = -1$$

$$\neq \qquad \neq$$

$$\operatorname{sgn}(0) = 0$$



$$\lim_{x\to a^+} f(x) = M \qquad \lim_{x\to a^-} f(x) = L$$

Theorem: Let f be a function defined on a neighborhood of a possibly except at a. Then

$\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ exist and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Example:

- $\displaystyle \lim_{x\to 5} \frac{|x-5|}{x^2-3x-10}$

$$\lim_{x\to 5} \frac{|\cancel{x-5}|}{(\cancel{x-5})(x+2)} = \lim_{x\to 5} \frac{1}{x+2}$$

not true

$$\lim_{x\to 5^+} \frac{|x-5|}{x^2-3x-10} = \lim_{x\to 5^+} \frac{|x-5|}{(x-5)(x+2)} = \lim_{x\to 5^+} \frac{\cancel{x-5}}{(\cancel{x-5})(x+2)} = \lim_{x\to 5^+} \frac{1}{x+2} = \frac{1}{7}$$

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \end{cases}$$

$$\lim_{x\to 5^-} \frac{|x-5|}{x^2-3x-10} = \lim_{x\to 5^-} \frac{|x-5|}{(x-5)(x+2)} = \lim_{x\to 5^-} \frac{-(\cancel{x-5})}{(\cancel{x-5})(x+2)} = \lim_{x\to 5^-} \frac{-1}{x+2} = \frac{-1}{7}$$

Because $\displaystyle \lim_{x\to 5^+} \frac{|x-5|}{x^2-3x-10} \neq \lim_{x\to 5^-} \frac{|x-5|}{x^2-3x-10}$, $\displaystyle \lim_{x\to 5} \frac{|x-5|}{x^2-3x-10}$ does not exist.

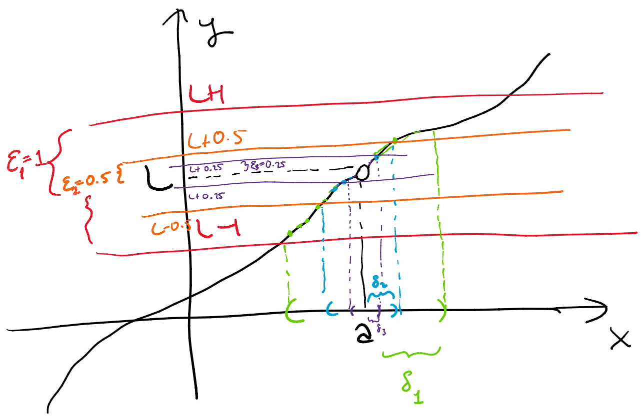
Because $\lim_{x\to 5^+} \frac{|x-5|}{x^2-3x-10} \neq \lim_{x\to 5^-} \frac{|x-5|}{x^2-3x-10}$, $\lim_{x\to 5} \frac{|x-5|}{x^2-3x-10}$ does not exist.

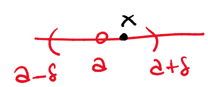
The formal definition of limit

Defn: Let f be a function defined on a neighborhood of a (possibly except at a). We say that the limit of f at a is L if

for every $\varepsilon>0$ there exist $\delta>0$ such that

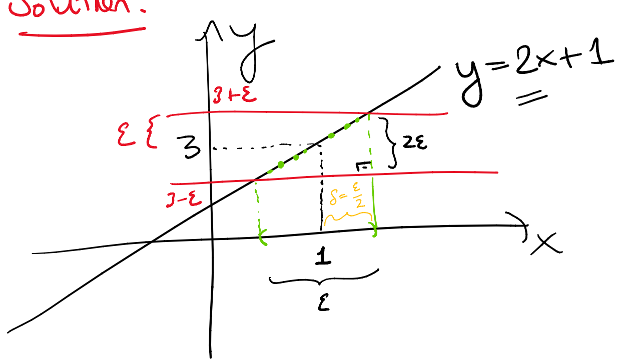
if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$





Example: Show that $\lim_{x\to 1} 2x+1 = 3$ (using the formal defn. of limit.)

Solution:



Given $\varepsilon>0$, choose $\delta = \frac{\varepsilon}{2} > 0$. Then

If $0<|x-1|<\delta$, then

$$|x-1|<\delta=\frac{\varepsilon}{2} \text{ and so}$$

$$|2x-2|<\varepsilon$$

$$|\underbrace{2x+1}_{f(x)} - \underbrace{3}_{L}| < \varepsilon$$

So $\lim_{x\to 1} 2x+1 = 3$

Example: Show that $\lim_{x\to 0} 2x\sin x = 0$

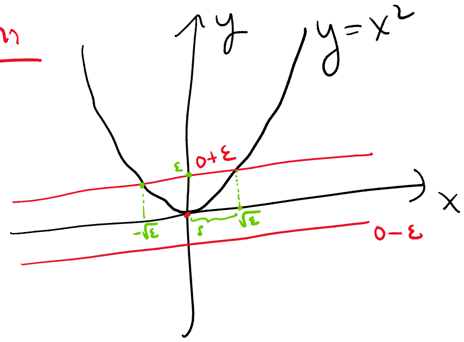
Solution: Given $\varepsilon>0$, choose $\delta=\frac{\varepsilon}{4}$. Then, if $0<|x-0|<\delta$, then

$$|x|<\delta \text{ and so}$$

$$|\underbrace{2x\sin x}_{f(x)} - \underbrace{0}_{L}| = |x|\,|2\sin x| \leq \delta\,|2\sin x| \leq 2\delta = 2\cdot\frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon$$

Example: Show that $\lim_{x \to 0} x^2 = 0$

Solution



Given $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon} > 0$.

Then if

$0 < |x-0| < \delta$, then

$$|x| < \delta = \sqrt{\varepsilon}$$

$$|x|^2 < \varepsilon$$

$$|\underbrace{x^2}_{f(x)} - \underbrace{0}_{L}| < \varepsilon$$

So $\lim_{x \to 0} x^2 = 0$.

Read more complicated examples in your book (e.g. $\lim_{x \to 2} x^2+1 = 5$)