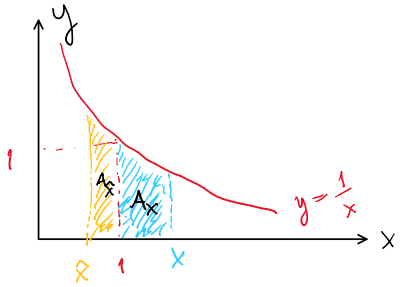
05 March 2020

Thursday, March 5, 2020   8:42 AM

The natural logarithm

For $x > 0$, we define the natural logarithm of x, shown by $\ln(x)$ as follows.



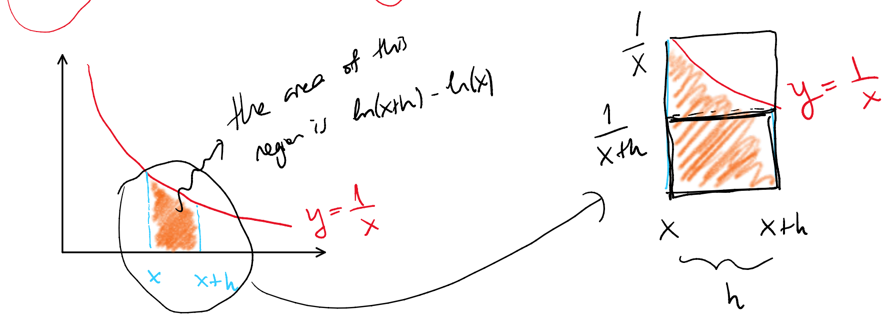
$$\ln(x) = \begin{cases} \text{the area of } A_x & \text{if } x \geq 1 \\ -\text{the area of } A_x & \text{if } x < 1 \end{cases}$$

Theorem: $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

Proof: Let $x > 0$ and $h > 0$. Then we have that

$$\frac{h}{x+h} < \ln(x+h) - \ln(x) < \frac{h}{x}$$

as can be seen from the picture. It follows that



$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}.$$

So, by the squeeze theorem, as $\lim\limits_{h \to 0^+} \frac{1}{x+h} = \lim\limits_{h \to 0^+} \frac{1}{x} = \frac{1}{x}$ we have

$$\lim_{h \to 0^+} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}.$$

One can similarly show $\lim\limits_{h \to 0^-} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}$

Thus $\frac{d}{dx}\ln(x) = \lim\limits_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}$

Properties of ln

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\ln(x^r) = r\ln(x)$
- $\ln(1) = 0$

will be defined soon

Proof of the first item: Let $f(x) = \ln(xy) - \ln(x)$. Then

$f'(x) = \frac{1}{xy} \cdot y - \frac{1}{x} = 0$ for all $x > 0$. Thus* $f(x) = c$

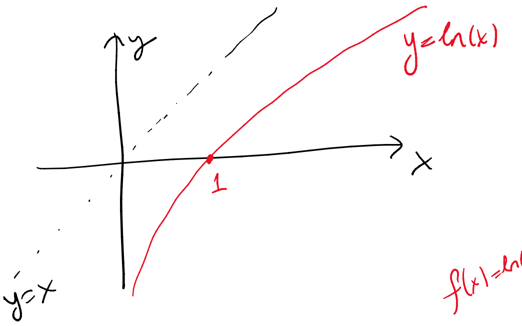
for all $x > 0$. Plugging in $x = 1$ gives $f(1) = \ln(y) - 0 = c$

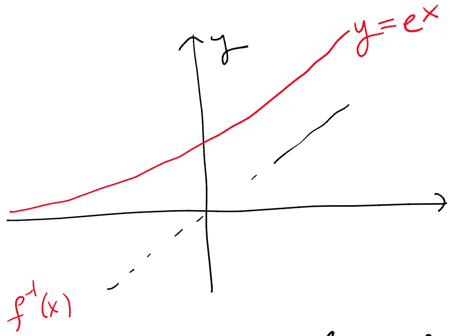
So $\underbrace{\ln(xy) - \ln(x)}_{f(x)} = \underbrace{\ln(y)}_{c}$

*: If $g'(x) = 0$ on an interval (a,b) and g is diff. on (a,b), then $g(x) = c$ for some constant on (a,b). This follows from MVT.

The exponential function:

We have $\frac{d}{dx}\ln(x) = \frac{1}{x} > 0$ for all $x > 0$. So $\ln$ is increasing on $(0,\infty)$ and so is one-to-one. Thus there is an inverse function of $\ln$ with domain $\mathbb{R}$ and range $(0,\infty)$. (Note that the domain of $\ln$ is $(0,\infty)$ and its range is $\mathbb{R}$.)





This inverse map is denoted by e^x. By the derivative formula for inverse functions, we get

$$\frac{d}{dx}(e^x) = \frac{d}{dx}(f^{-1})(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\frac{1}{f^{-1}(x)}} = f^{-1}(x) = e^x$$

Properties of e^x

- $e^{x+y} = e^x \cdot e^y$
- $e^{x-y} = e^x / e^y$
- $(e^x)^r = e^{xr}$

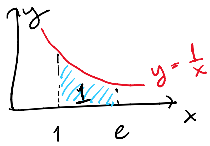
The number e

The value of the exp. function at $x=1$ is a special number called, e.

$$e = e^1 = 2.718281828\ldots$$

$$= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$



Keep in mind the following facts:

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} \ln(x) = +\infty$$

$$\lim_{x\to+\infty} e^x = \infty \qquad \lim_{x\to 0^+} \ln(x) = -\infty$$

General exponential functions and logarithms

For $a > 0$, for all x in $\mathbb{R}$, we define the exponential function with base a to be

$$a^x = e^{x\ln(a)}$$

The logarithm with base a is defined to be the inverse of a^x.

$\log_a(x)$ is the inverse map of a^x

Properties of general exponential maps and logarithms

Let $a>0$. Then

- $a^{x+y} = a^x . a^y$
- $a^{x-y} = a^x / a^y$
- $(a^x)^r = a^{xr}$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r \log_a(x)$
- $\log_a x = \dfrac{\log_b x}{\log_b a}$ (for $b>0$)

- $\dfrac{d}{dx}(a^x) = \dfrac{d}{dx}\left(e^{x\ln(a)}\right) = e^{x\ln(a)} . \ln(a) = \ln(a) . a^x$

- $\dfrac{d}{dx}(\log_a x) = \dfrac{d}{dx}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln a} \dfrac{1}{x}$

Example: Find the derivative of x^x (for $x>0$.)

Solution: $\dfrac{d}{dx}(x^x) = \dfrac{d}{dx}(e^{x\ln x}) = e^{x\ln x} . (1.\ln x + x.\frac{1}{x}) = x^x . (\ln x + 1)$

Example: Find the derivative of $f(x) = \sin x^{\cos x}$

Solution: We can do the exact same trick as before, that is, $\dfrac{d}{dx}(\sin x^{\cos x}) = \dfrac{d}{dx}(e^{\cos x \ln \sin x}) = \ldots$

or we set $y = \sin x^{\cos x}$ and then take natural log. So we get

this trick is called "Logarithmic differentiation"

$$\ln y = \ln(\sin x^{\cos x}) = \cos x \ln(\sin x)$$

$$\frac{1}{y} . y' = -\sin x . \ln \sin x + \cos x . \frac{1}{\sin x} . \cos x \quad \text{and so} \quad y' = f'(x) = y . (-\sin x \ln \sin x + \cot x \cos x)$$

$$= \sin x^{\cos x} (-\sin x \ln \sin x + \cot x \cos x)$$

Example: Assuming that the equation $x^y + y^x = 8$ defines y as a diff. function of x at $(2,2)$ find the equation of the tangent line to the curve $x^y + y^x = 8$ at $(2,2)$.

Solution: Differentiating both sides gives,

$$x^y + y^x = 8$$

$$e^{y\ln x} + e^{x\ln y} = 8$$

$$e^{y\ln x} . (y' . \ln x + y\frac{1}{x}) + e^{x\ln y} . (1.\ln y + x\frac{1}{y} . y') = 0$$

Plugging in $(x,y) = (2,2)$ gives,

$$\cancel{4}(y'|_{(2,2)} . \ln(2) + 1) + \cancel{4}(\ln 2 + y'|_{(2,2)}) = 0 \quad \text{and so} \quad y'|_{(2,2)} = \frac{-\ln 2 - 1}{\ln 2 + 1} = -1$$

So an equation of the tangent line to this curve at $(2,2)$ is

$$y-2=(-1)\cdot(x-2)$$
$$y=-x+4$$

Example: Find y' where $y = 2^x \cdot \dfrac{(x+1)(x-1)(x+2)}{\sin x \cdot \cos x}$

Solution: We have that

$$\ln y = \ln(2^x) + \ln\left(\frac{(x+1)(x-1)(x+2)}{\sin x \cos x}\right)$$

$$= \underbrace{\ln(2^x)}_{x\ln 2} + \left[\ln(x+1) + \ln(x-1) + \ln(x+2) - \ln(\sin x) - \ln(\cos x)\right]$$

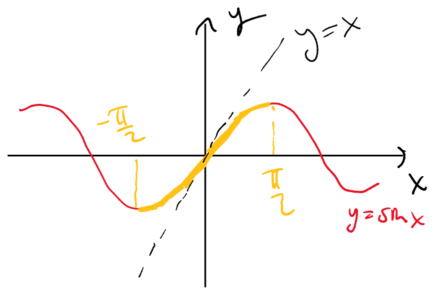
differentiate

$$y \cdot \frac{y'}{y} = \left(\ln 2 + \left[\frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{x+2} - \frac{1}{\sin x}\cos x - \frac{1}{\cos x}(-\sin x)\right]\right)\cdot y$$

$$y' = \left[\frac{2^x(x+1)(x-1)(x+2)}{\sin x \cos x}\right]\left(\ln 2 + \frac{1}{x+1} - \frac{1}{x-1} + \frac{1}{x+2} - \cot x + \tan x\right)$$

Inverse trigonometric functions

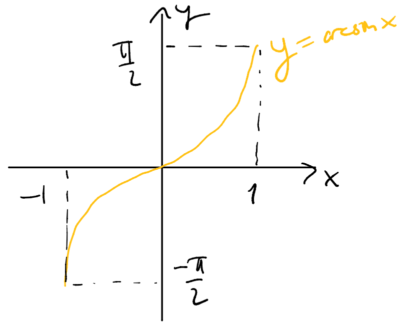
The function sine is <u>not</u> one-to-one on its domain $\mathbb{R}$. However, if we restrict its domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it becomes one-to-one



$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$$

The inverse of this function is called the arcsine function

$$\arcsin : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



Note that

$$\sin(\arcsin(x)) = x \quad \text{for all } x \text{ in } [-1,1]$$
$$\arcsin(\sin(x)) = x \quad \text{for all } x \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

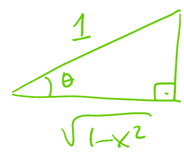
However, in general, $\arcsin(\sin(x)) \neq x$ for arbitrary x.

$$\arcsin\left(\sin\left(\frac{5\pi}{2}\right)\right) = \arcsin(1) = \frac{\pi}{2} \neq \frac{5\pi}{2}$$

We have $\sin(\arcsin(x)) = x$

$$\cos(\arcsin(x)) \cdot \frac{d}{dx}\arcsin x = 1$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$



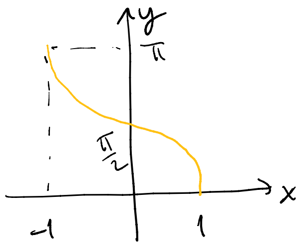
$\sin\theta = x$

$\theta = \arcsin x$

$\cos\theta = \cos(\arcsin(x))$

$= \sqrt{1-x^2}$

By restricting the domain of cosine to $[0, \pi]$, one can similarly define the arccosine function



$$\arccos : [-1, 1] \to [0, \pi]$$

We have that, for all x in $(-1,1)$,

$$\cos(\arccos x) = x$$

$$-\sin(\arccos x) \cdot \frac{d}{dx} \arccos x = 1$$

use a right triangle as before

$$\frac{d}{dx} \arccos x = \frac{-1}{\sin(\arccos x)} = \frac{-1}{\sqrt{1-x^2}}$$