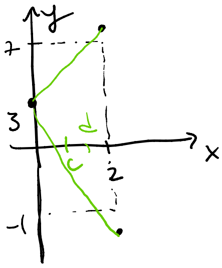
03 March 2020

Tuesday, March 3, 2020 3:45 PM

Example: Let f be a differentiable function on $\mathbb{R}$ such that $|f'(x)| \leq 2$ and $f(0) = 3$. Show that $-1 \leq f(2) \leq 7$.

Solution:



Suppose that $-1 \leq f(2) \leq 7$ is not true.

Case I ($f(2) > 7$): Then, by MVT, there is $0 < c < 2$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) - 3}{2}$$

As $f(2) > 7$, we get $\frac{f(2)-3}{2} = f'(c) > 2$ which contradicts the given assumption.

Case II ($f(2) < -1$): Then, by MVT, there is $0 < d < 2$ such that $f'(d) = \frac{f(2) - f(0)}{2-0}$

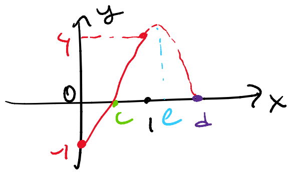
As $f(2) < -1$, we get $\frac{f(2)-3}{2} = f'(c) < -2$ which is a contradiction.

Thus we must have

Example: Show that the equation $x^3 + 4x - 1 = 0$ has exactly one solution.

Solution: Let $f(x) = x^3 + 4x - 1$. Then f is cont. and differentiable everywhere.

Since $f(0) = -1 < 0 < 4 = f(1)$ and f is cont on $[0,1]$, by IVT,



there is some $0 < c < 1$ such that $f(c) = 0$

So $c^3 + 4c - 1 = 0$

Note that $f'(x) = 3x^2 + 4 > 0$ for all x.

Solution I

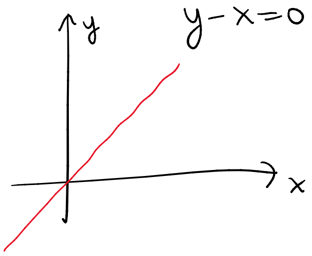
If it were the case that there is another $d \neq c$ with $f(d) = 0$, by MVT (or Rolle's theorem) applied to $[c,d]$ we would have

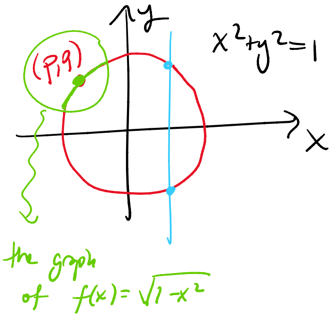
$$f'(e) = \frac{f(c) - f(d)}{c - d} = \frac{0 - 0}{c - d} = 0 > 0$$

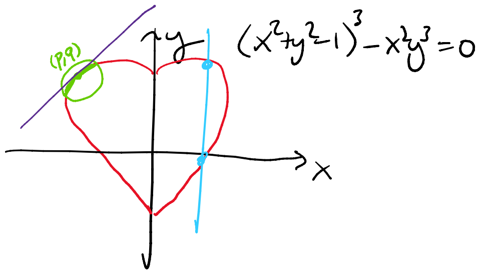
for some e between c and d. Thus there is only one point c with $f(c) = 0$.

Solution II As $f'(x) > 0$ for all x, f is increasing. Thus f is one-to-one which means that it cannot have distinct c, d with $f(c) = f(d) = 0$

Implicit differentiation Given an equation $F(x,y)=0$ (which usually defines a curve in plane) and a point (p,q) with $F(p,q)=0$, it is possible that "around" (p,q) the equation $F(x,y)=0$ $\boxed{\text{defines } y \text{ as a differentiable of } x}$. In this case, $\frac{dy}{dx}$ can be found by differentiating both sides of $F(x,y)=0$ pretending that y is a function of x.

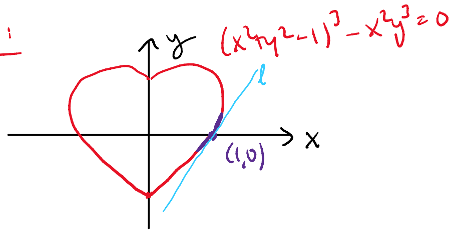






Example: Assuming that $(x^2+y^2-1)^3-x^2y^3=0$ defines y as a diff. function of x around $(1,0)$, find the equation of the tangent line to this curve at $(1,0)$.

Solution:



We have $(x^2+y^2-1)^3=x^2y^3$ and so

$x^2+y^2-1=x^{\frac{2}{3}}y$ and differentiating both sides gives

$2x+2y\frac{dy}{dx}=\frac{2}{3}x^{\frac{1}{3}}\cdot y+x^{\frac{2}{3}}\cdot\frac{dy}{dx}$. Plugging in $(x,y)=(1,0)$

we get,

$2=1\cdot\left.\frac{dy}{dx}\right|_{(1,0)}$ and so $\left.\frac{dy}{dx}\right|_{(1,0)}=2$

Thus an equation for l is

$$y-0=2(x-1)$$
$$y=2x-2$$

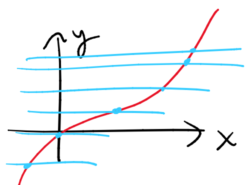
Inverse functions and their derivatives

A function f is said to be **one-to-one** if $x_1\neq x_2$ implies $f(x_1)\neq f(x_2)$.

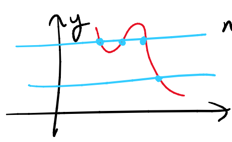
(equivalently, $f(x_1)=f(x_2)$ implies $x_1=x_2$)

Examples:

- $f(x)=x^2$ not one-to-one as $-1\neq 1$ but $f(-1)=f(1)=1$
- $f(x)=\sqrt[3]{x}$ one-to-one because if $f(x)=f(\tilde{x})$ then $\sqrt[3]{x}=\sqrt[3]{\tilde{x}}$ and so $x=\tilde{x}$.



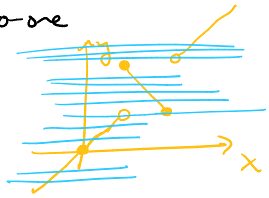
One-to-one



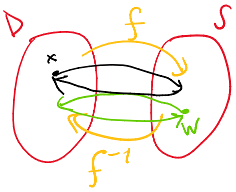
not one-to-one

Fact: • If f is increasing, then f is one-to-one
• If f is decreasing, then f is one-to-one

In general one-to-one $\not\Rightarrow$ increasing / decreasing



Let f be a one-to-one function with domain D and range S.



Then there is a function with domain S and range D, which will be denoted by f^{-1}, such that

$$f(f^{-1}(w)) = w \quad \text{for all } w \text{ in } S$$
$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in } D$$

This function is called **the inverse of f** and is unique.

$$\boxed{f(x) = y \quad \text{iff} \quad f^{-1}(y) = x}$$

how?

You can set $f(x)=y$ and try to solve for x in terms of y

$y = x-2$
$x = y+2$

and replace x by $f^{-1}(x)$ and y by x

$f^{-1}(x) = x-2$

Example: • Let $f(x) = x-2$. Then $f^{-1}(x) = x+2$

$$f(f^{-1}(x)) = f(x+2) = (x+2)-2 = x$$
$$f^{-1}(f(x)) = f^{-1}(x-2) = (x-2)+x = x$$

• Let $f(x) = \frac{x+1}{x-1}$. Then $f^{-1}(x) = \frac{x+1}{x-1} = f(x)$

$y = \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$

$y-1 = \frac{2}{x-1}$

$x-1 = \frac{2}{y-1}$

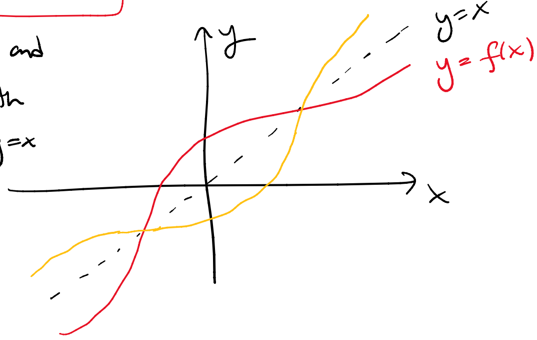
$x = 1 + \frac{2}{y-1} = \frac{y+1}{y-1}$

• Let $f(x) = x + \tan x$. Then ... !! ... $f^{-1}(x) = ??$

$y = x + \tan x$

!!??

Fact: The graphs of f and f^{-1} are symmetric with respect to the line $y=x$



Derivative of the inverse functions

Suppose that f^{-1} is differentiable at x and f is differentiable at $f^{-1}(x)$. Then

$$f(f^{-1}(x)) = x$$

differentiate wrt x

$$f'(f^{-1}(x))\cdot(f^{-1})'(x) = 1$$

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

Example: Let $f(x) = 2x + \sin x$. Show that f is one-to-one and find $(f^{-1})'(\pi+1)$.

Solution: Note that $f'(x) = 2 + \cos x > 0$ for all x and so f is increasing. Thus f is one-to-one. So f^{-1} exists. (Finding f^{-1} explicitly is "impossible".)

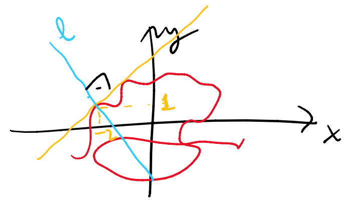
By the formula we derived, we have

$$(f^{-1})'(\pi+1) = \frac{1}{f'(f^{-1}(\pi+1))} = \frac{1}{f'(\frac{\pi}{2})} = \frac{1}{2}$$

$f^{-1}(\pi+1) = \frac{\pi}{2}$ because $f(\frac{\pi}{2}) = \pi+1$

Example: Assuming that the equation $x^2 + xy + 2y^3 = 4$ defines y as a differentiable function of x around $(-2,1)$, find the normal line to the curve defined by this equation at $(-2,1)$.

Solution:



Diff. both sides of the eq. gives

$$2x + \left(1\cdot y + x\frac{dy}{dx}\right) + 6y^2\cdot\frac{dy}{dx} = 0$$

Plugging in $(x,y) = (-2,1)$ gives

$$\left.\frac{dy}{dx}\right|_{(-2,1)} = \frac{3}{4}$$

Thus the slope of the normal line is $\frac{-4}{3}$. So its equation is

$$y - 1 = \frac{-4}{3}(x-2).$$