Problem 1. [Ch. 4 Problem 5] Let X and Y be independent random variables, uniformly distributed in the interval $[0, 1]$. Find the CDF and the PDF of $|\mathsf{X} - \mathsf{Y}|$.

Problem 2. Let Y be $\mathsf{Y} = \sum_{t=1}^{n} \mathsf{X}_t$, where $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are independent identically distributed continuous random variables with PDF f_{X}, i.e.,

$$f_{\mathsf{X}_1,\ldots,\mathsf{X}_n}(x_1, \ldots, x_n) = \prod\nolimits_{t=1}^{n} f_{\mathsf{X}}(x_t) \,.$$

What is $\mathbf{E}[\mathsf{X}_1 | \mathsf{Y}]$? (*Hint:* Can you answer the question for $n = 2$ case?)

Problem 3. Let Z be $\mathsf{Z} = \mathsf{X}_{\mathsf{T}}$ where $\mathsf{X}_1, \ldots, \mathsf{X}_n$ and T be independent random random variables and T be an integer valued positive random variable satisfying $\mathbf{P}(\mathsf{T} \leq n) = 1$. Show that

$$M_{\mathsf{Z}}(s) = \sum\nolimits_{t=1}^{n} p_{\mathsf{T}}(t)\, M_{\mathsf{X}_t}(s) \tag{1}$$

Problem 4. Let the random variable X be $\mathsf{X} = \mathsf{Y}/\mathsf{Z}$, where Y and Z are independent, Y is an exponentially distributed random variable with parameter λ and Z be a random variable uniformly distributed on $[a, b]$ for $b > a > 0$, i.e.,

$$f_{\mathsf{Y}}(y) = \begin{cases} \lambda e^{-\lambda y}, & \text{if } y \geq 0, \\ 0, & \text{if } y < 0, \end{cases} \qquad \text{and} \qquad f_{\mathsf{Z}}(z) = \begin{cases} \frac{1}{b-a}, & \text{if } z \in [a, b], \\ 0, & \text{if } z \notin [a, b], \end{cases}.$$

What is the moment generating function M_{X} of the random variable X?

Problem 5. For each $t \in \{1, \ldots, n\}$ let Y_t be an exponentially distributed random variable with parameter λ that is independent of X_{t-1} and the conditional PMF of X_t given X_{t-1} and Y_t be

$$p_{\mathsf{X}_t|\mathsf{X}_{t-1},\mathsf{Y}_t}(x_t|x_{t-1}, y_t) = \begin{cases} 1/2, & \text{if } x_t = \frac{x_{t-1}}{2} + y_t, \\ 1/2, & \text{if } x_t = \frac{x_{t-1}}{2}, \end{cases}.$$

Determine the moment generating function M_{X_n} of the random variable X_n in terms of the moment generating function M_{X_0} of the random variable X_0.

Problem 6. Let W be

$$\mathsf{W} = \frac{\mathsf{X}+\mathsf{Y}\mathsf{Z}}{\sqrt{1+\mathsf{Z}^2}}, \tag{2}$$

where X, Y, and Z independent identically distributed zero mean Gaussian random variables with variance σ^2. What is f_{S}? (*Hint:* You need not to do tedious calculations to solve this problem.)

Problem 7. Let Z be

$$\mathsf{Z} = \mathsf{X} + \mathsf{Y}, \tag{3}$$

where X is exponentially distributed with parameter λ and Y is a mean β random variable independent of X. If Z is exponentially distributed, then what is F_{Y}? (*Hint:* You might want to invoke (1) at some point.)

Solution of Problem 1 [Ch. 4 Problem 5].

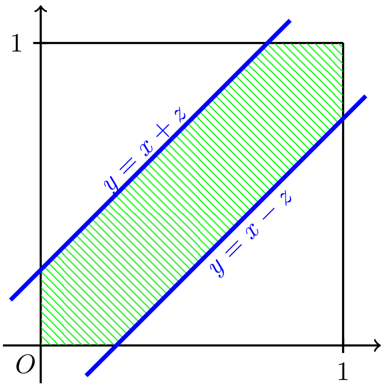

Figure 1. $y = x + z$ and $y = x - z$ lines and the region on which $|x - y| \leq z$ and $f_{\mathsf{X},\mathsf{Y}}(x, y) > 0$ for a $z \in [0, 1]$.

Let the random variable Z be

$$\mathsf{Z} = |\mathsf{X} - \mathsf{Y}|. \tag{4}$$

Recall that $F_{\mathsf{Z}}(z) = \mathbf{P}(\mathsf{Z} \leq z)$ by definition. Thus using (4), we get

$$F_{\mathsf{Z}}(z) = \mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z). \tag{5}$$

Since the absolute value is always non-negative, $\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z) = 0$ for all $z < 0$. In order to determine $\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z)$ for $z \geq 0$ case, we can use $f_{\mathsf{X},\mathsf{Y}}$

$$\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z) \int_{-\infty}^{\infty} \int_{x-z}^{x+z} f_{\mathsf{X},\mathsf{Y}}(x, y) \, \mathrm{d}y \mathrm{d}x \tag{6}$$

Thus we determine $f_{\mathsf{X},\mathsf{Y}}$, first. To that end note that the independence of X and Y imply

$$f_{\mathsf{X},\mathsf{Y}}(x, y) = f_{\mathsf{X}}(x) \, f_{\mathsf{Y}}(y).$$

On the other hand both X and Y are both uniformly distributed in the interval $[0, 1]$, i.e.,

$$f_{\mathsf{X}}(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_{\mathsf{Y}}(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$f_{\mathsf{X},\mathsf{Y}}(x, y) = \begin{cases} 1 & x \in [0,1] \text{ and } y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Then $f_{\mathsf{X},\mathsf{Y}}(x, y)$ is positive only for x and y values satisfying $|x - y| \leq 1$ by (6). Thus $\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z) = 1$ for all $z \geq 1$. In order to calculate $\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z)$ for $z \in [0, 1)$ we can use (6) and (7) with the help of Figure 1. Note that in integral given (6) is the area between the two blue lines in Figure 1. In that are the $f_{\mathsf{X},\mathsf{Y}}(x, y)$ is positive only for the shaded region and its value is one in the shaded region. Thus $\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z)$ is equal to the area of the shaded region. We can calculate that area by subtracting the areas of the triangle to the lower-right and upper-left of the shaded region from the area of the unit square. The area of each triangle is $\frac{1}{2}(1 - z)^2$. Thus $\mathbf{P}(|\mathsf{X} - \mathsf{Y}| \leq z) = 1 - (1 - z)^2$ for $z \in [0, 1)$. Thus

$$F_{\mathsf{Z}}(z) = \begin{cases} 0, & z < 0, \\ 1 - (1 - z)^2, & z \in [0, 1], \\ 1 & z > 1, \end{cases}.$$

Then using $f_{\mathsf{Z}}(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_{\mathsf{Z}}(z)$, we get

$$f_{\mathsf{Z}}(z) = \begin{cases} 2(1 - z), & z \in [0, 1], \\ 0 & \text{otherwise}, \end{cases}.$$

$\square$

Solution of Problem 2. Note that as a result of the symmetry of the problem we have

$$\mathbf{E}[\mathsf{X}_1 | \mathsf{Y}] = \mathbf{E}[\mathsf{X}_t | \mathsf{Y}] \qquad \forall t \in \{2, \ldots, n\}. \tag{8}$$

In fact a stronger result holds for conditional PDFs: $f_{\mathsf{X}_1|\mathsf{Y}}(x|y) = f_{\mathsf{X}_t|\mathsf{Y}}(x|y)$ for all x, y, and $t \in \{2, \ldots, n\}$.

On the other hand, $\mathbf{E}[\mathsf{Y} | \mathsf{Y}] = \mathsf{Y}$ and $\mathsf{Y} = \sum_{t=1}^{n} \mathsf{X}_t$ imply

$$\mathsf{Y} = \sum\nolimits_{t=1}^{n} \mathbf{E}[\mathsf{X}_t | \mathsf{Y}]. \tag{9}$$

Using (8) and (9) we get $\mathbf{E}[\mathsf{X}_1 | \mathsf{Y}] = \frac{\mathsf{Y}}{n}$.

Remark 1. Note that our reasoning has nothing to do with the existence of the probability density functions, f_{X_t}'s. Hence $\mathbf{E}[\mathsf{X}_1 | \mathsf{Y}] = \frac{\mathsf{Y}}{n}$ for any independent identically distributed $\mathsf{X}_1, \ldots, \mathsf{X}_n$, not just the continuous ones.

$\square$

Solution of Problem 3.

$$\begin{aligned}
M_{\mathsf{Z}}(s) &= \mathbf{E}\left[\mathsf{e}^{\mathsf{sX_T}}\right] && \text{by the defintion of moment generating function and the random variable Z,}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{sX_T}}\,\middle|\,\mathsf{T}\right]\right] && \text{by the law of iterated expectations,}\\
&= \mathbf{E}[\mathsf{M}_{\mathsf{X_T}}(s)] && \text{by the defintion of moment generating function and independence,}\\
&= \sum\nolimits_{t=1}^{n} p_{\mathsf{T}}(t)\, M_{\mathsf{X}_t}(s)
\end{aligned}$$

□

Solution of Problem 4. One can first calculate the PDF f_{X} the random variable X and then determine the moment generating function M_{X}. However, the use of conditional expectations provides us a less tedious way to solve this problem.

$$\begin{aligned}
M_{\mathsf{X}}(s) &= \mathbf{E}\left[\mathsf{e}^{\mathsf{sX}}\right] && \text{by the defintion of moment generating function,}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{sX}}\,\middle|\,\mathsf{Z}\right]\right] && \text{by the law of iterated expectations,}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{s}\frac{\mathsf{Y}}{\mathsf{Z}}}\,\middle|\,\mathsf{Z}\right]\right] && \text{because } \mathsf{X} = {}^{\mathsf{Y}}\!/\mathsf{z},\\
&= \mathbf{E}\left[\mathsf{M}_{\mathsf{Y}}\left(\tfrac{s}{\mathsf{Z}}\right)\right] && \text{because } \mathbf{E}\left[\mathsf{e}^{\frac{\mathsf{s}}{\mathsf{Z}}\mathsf{Y}}\,\middle|\,\mathsf{Z}\right] = M_{\mathsf{Y}}\left(\tfrac{s}{\mathsf{Z}}\right) \text{ by the defintion of moment generating function,}\\
&= \mathbf{E}\left[1+\tfrac{\mathsf{s}}{\mathsf{Z}\lambda-\mathsf{s}}\right] && \text{because } M_{\mathsf{Y}}(s) = \tfrac{\lambda}{\lambda-s}\text{, see page 239 of the textbook,}\\
&= 1+\tfrac{s}{\lambda}\int_a^b \tfrac{1}{b-a}\,\tfrac{1}{z-\frac{s}{\lambda}}\,\mathrm{d}z && \text{because Z is uniformly distributed on } [a,b],\\
&= 1+\tfrac{s}{\lambda}\tfrac{1}{b-a}\ln\left(z-\tfrac{s}{\lambda}\right)\Big|_a^b\\
&= 1+\tfrac{s}{\lambda(b-a)}\ln\left(\tfrac{\lambda b-s}{\lambda a-s}\right)
\end{aligned}$$

□

Solution of Problem 5. One can in principle use the inversion property to determine the CDF of the random variable X_0 and then determine first the CDF of the random variable X_n and then the moment generating function $M_{\mathsf{X_n}}$. However, as was the case in the previous example the conditional expectations provides us a much simpler way to solve this problem.

$$\begin{aligned}
M_{\mathsf{X}}(s) &= \mathbf{E}\left[\mathsf{e}^{\mathsf{sX_n}}\right] && \text{by the defintion of moment generating function,}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{sX_n}}\,\middle|\,\mathsf{X_{n-1}},\mathsf{Y_n}\right]\right] && \text{by the law of iterated expectations,}\\
&= \mathbf{E}\left[\tfrac{1}{2}\mathsf{e}^{\mathsf{s}\left(\frac{\mathsf{X_{n-1}}}{2}+\mathsf{Y_n}\right)}+\tfrac{1}{2}\mathsf{e}^{\mathsf{s}\frac{\mathsf{X_{n-1}}}{2}}\right] && \text{by invoking the expression for } p_{\mathsf{X_n}|\mathsf{X_{n-1}},\mathsf{Y_n}} \text{ to calculate } \mathbf{E}\left[\mathsf{e}^{\mathsf{sX_n}}\,\middle|\,\mathsf{X_{n-1}},\mathsf{Y_n}\right],\\
&= \mathbf{E}\left[\tfrac{1}{2}\left(1+\mathsf{e}^{\mathsf{sY_n}}\right)\mathsf{e}^{\mathsf{s}\frac{\mathsf{X_{n-1}}}{2}}\right]\\
&= \mathbf{E}\left[\tfrac{1}{2}\left(1+\mathsf{e}^{\mathsf{sY_n}}\right)\right]\mathbf{E}\left[\mathsf{e}^{\mathsf{s}\frac{\mathsf{X_{n-1}}}{2}}\right] && \text{beause } \mathsf{Y}_n \text{ and } \mathsf{X}_{n-1} \text{ are independent,}\\
&= \tfrac{1}{2}\left[1+M_{\mathsf{Y_n}}(s)\right]M_{\mathsf{X_{n-1}}}\left(\tfrac{\mathsf{s}}{2}\right) && \text{by the defintion of moment generating function,}\\
&= \tfrac{\lambda-s/2}{\lambda-s}M_{\mathsf{X_{n-1}}}\left(\tfrac{\mathsf{s}}{2}\right) && \text{because } M_{\mathsf{Y_n}}(s) = \tfrac{\lambda}{\lambda-s}\text{, see page 239 of the textbook,}\\
&= \tfrac{\lambda-s/2^2}{\lambda-s}M_{\mathsf{X_{n-2}}}\left(\tfrac{\mathsf{s}}{2^2}\right) && \text{because } M_{\mathsf{X_{n-1}}}(s) = \tfrac{\lambda-s/2}{\lambda-s}M_{\mathsf{X_{n-2}}}\left(\tfrac{\mathsf{s}}{2}\right) \text{ as a result of analgous arguments,}\\
&= \tfrac{\lambda-s/2^n}{\lambda-s}M_{\mathsf{X_0}}\left(\tfrac{\mathsf{s}}{2^{\mathsf{n}}}\right)
\end{aligned}$$

□

Solution of Problem 6. The critical step in solving this problem is considering the behavior of W conditioned on Z. Note that as a result of independence of X, Y, and Z we have

$$f_{\mathsf{X,Y|Z}}(x,y|z) = f_{\mathsf{X}}(x)\, f_{\mathsf{Y}}(y)\,, \tag{10}$$

Hence for all z, the random variables X and Y are independent zero mean Gaussian random variables conditioned on the event $\mathsf{Z} = z$, as well. We can use this observation either via the moment generating function or by using PDFs to obtain the result.

- Let us first use the Moment generating function

$$
\begin{aligned}
M_{\mathsf{W}}(s) &= \mathbf{E}\left[\mathsf{e}^{\mathsf{sW}}\right] && \text{by the defintion of moment generating function,}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{sW}}\middle|\,\mathsf{Z}\right]\right] && \text{by the law of iterated expectations.}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{s}\frac{\mathsf{X}+\mathsf{YZ}}{\sqrt{1+\mathsf{Z}^2}}}\middle|\,\mathsf{Z}\right]\right] && \text{by (2)}\\
&= \mathbf{E}\left[\mathbf{E}\left[\mathsf{e}^{\mathsf{s}\frac{\mathsf{X}}{\sqrt{1+\mathsf{Z}^2}}}\middle|\,\mathsf{Z}\right]\mathbf{E}\left[\mathsf{e}^{\mathsf{s}\frac{\mathsf{YZ}}{\sqrt{1+\mathsf{Z}^2}}}\middle|\,\mathsf{Z}\right]\right] && \text{by (10)}\\
&= \mathbf{E}\left[\mathsf{e}^{\frac{\mathsf{s}^2}{2}\frac{\sigma^2}{1+\mathsf{Z}^2}}\mathsf{e}^{\frac{\mathsf{s}^2}{2}\frac{\mathsf{Z}^2}{1+\mathsf{Z}^2}}\right] && \text{because}\begin{cases}\mathsf{A}\sim\mathcal{N}(0,\sigma^2)\Leftrightarrow M_{\mathsf{A}}(s)=e^{\frac{s^2}{2}\sigma^2}, \text{ see page 239 of the textbook,}\\ \mathbf{E}[\mathsf{A}g(\mathsf{Z})|\,\mathsf{Z}]=g(\mathsf{Z})\mathbf{E}[\mathsf{A}|\,\mathsf{Z}] \text{ for any function } g \text{ and rand. var. } \mathsf{A}.\end{cases}\\
&= \mathbf{E}\left[\mathsf{e}^{\frac{\mathsf{s}^2}{2}\sigma^2}\right]\\
&= e^{\frac{s^2}{2}\sigma^2}
\end{aligned}
$$

  Then $\mathsf{W}\sim\mathcal{N}(0,\sigma^2)$ by the inversion property, see page 234 of the textbook. Thus $f_{\mathsf{W}}(w)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{w^2}{2\sigma^2}}$.
- Note that $a\mathsf{X}+b\mathsf{Y}\sim\mathcal{N}(a\mu_{\mathsf{X}}+b\mu_{\mathsf{Y}},a^2\sigma_{\mathsf{X}}^2+b^2\sigma_{\mathsf{Y}}^2)$ for any two constants $a,b\in\mathbb{R}$ for any two independent random variables $\mathsf{X}\sim\mathcal{N}(\mu_{\mathsf{X}},\sigma_{\mathsf{X}}^2)$ and $\mathsf{Y}\sim\mathcal{N}(\mu_{\mathsf{Y}},\sigma_{\mathsf{Y}}^2)$. Note that this is exactly the situation we have for W conditioned on the event $\mathsf{Z}=z$, for $a=\frac{1}{\sqrt{1+z^2}}$, $b=\frac{1}{\sqrt{1+z^2}}$, $\mu_{\mathsf{X}}=0$, $\mu_{\mathsf{Y}}=0$, $\sigma_{\mathsf{X}}=\sigma$, and $\sigma_{\mathsf{Y}}=\sigma$ as a result of (10). Thus,

$$f_{\mathsf{W}|\mathsf{Z}}(w|z)=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{w^2}{2\sigma^2}}\mathrm{d}\tau$$

  Then $f_{\mathsf{W}}(w)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{w^2}{2\sigma^2}}$ because $f_{\mathsf{W}}(w)=\mathbf{E}\left[\mathsf{f}_{\mathsf{W}|\mathsf{Z}}(w|\mathsf{Z})\right]$.

Remark 2. Note that the distribution of the random variable Z played no role in our calculations. Thus $f_{\mathsf{W}}(w)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{w^2}{2\sigma^2}}$ holds as long as (2) holds for independent random variables X, Y, and Z, where X and Y are identically distributed zero mean Gaussian random variables with variance σ^2.

□

Solution of Problem 7. $\mathbf{E}[\mathsf{X}]=\frac{1}{\lambda}$, see page 182 of the textbook. On the other hand $\mathbf{E}[\mathsf{Z}]=\mathbf{E}[\mathsf{X}]+\mathbf{E}[\mathsf{Y}]$ thus $\mathbf{E}[\mathsf{Z}]=\frac{1}{\lambda}+\beta$. On the other hand Z is exponentially distributed, thus the parameter of its exponential distribution is $\frac{1}{\mathbf{E}[\mathsf{Z}]}=\frac{\lambda}{1+\lambda\beta}$. Hence, we can determine the moment generating function of Z using the fact that it is exponentially distributed, see page 239 of the textbook,

$$M_{\mathsf{Z}}(s)=\frac{\lambda}{\lambda-s(1+\lambda\beta)} \tag{11}$$

Similarly, the moment generating function of Xis

$$M_{\mathsf{X}}(s)=\frac{\lambda}{\lambda-s} \tag{12}$$

On the other hand, since X and Y are independent moment generating function of their sum is equal to product of moment generating functions of each:

$$M_{\mathsf{Z}}(s)=M_{\mathsf{X}}(s)\,M_{\mathsf{Y}}(s) \tag{13}$$

Using (11), (12), and (13), we can determine the moment generating function of Y:

$$
\begin{aligned}
M_{\mathsf{Y}}(s) &= \frac{\lambda-s}{\lambda-s(1+\lambda\beta)}\\
&= \frac{1}{1+\lambda\beta}+\frac{\lambda\beta}{1+\lambda\beta}\frac{\lambda}{\lambda-s(1+\lambda\beta)}
\end{aligned}
$$

Note that $\frac{\lambda}{\lambda-s(1+\lambda\beta)}$ is the moment generating function of an exponentially distributed random variable with parameter $\frac{\lambda}{1+\lambda\beta}$ and 1 is the moment generating function of the random variable which is equal to 0 with probability 1. Thus using (1) and the inversion property we can conclude that

$$F_{\mathsf{Y}}(y)=\begin{cases}0 & \text{if } y<0\\ 1-\frac{\lambda\beta}{1+\lambda\beta}e^{-\frac{\lambda}{1+\lambda\beta}y} & \text{if } y\geq 0\end{cases} \tag{14}$$

□